11th Seminar for Homogenization and Quality Control in Climatological Databases and 6th Interpolation Conference jointly organized with the 14th EUMETNET Data Management Workshop

A statistical method for the attribution of change-points in segmentation of IWV difference time series

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9 May, 2023

Introduction

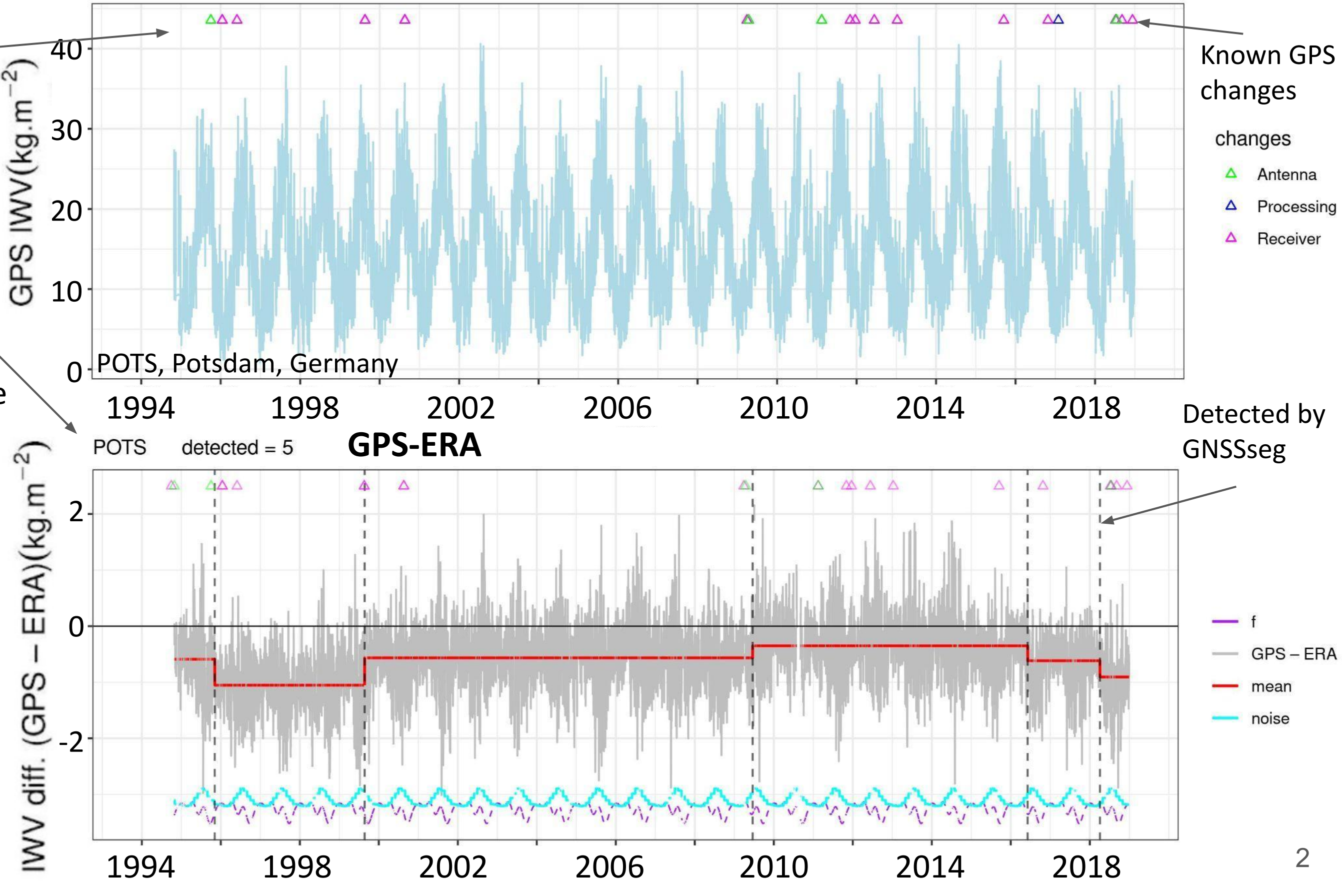
1. GPS IWV series contains known equipment changes, but it's hard to see any induced IWV changes

2. Differenced series (GPS-ERA) is segmented using the statistical method (GNSSseg package)*

3. Some detected change points are "close" to known equipment changes and others are not…

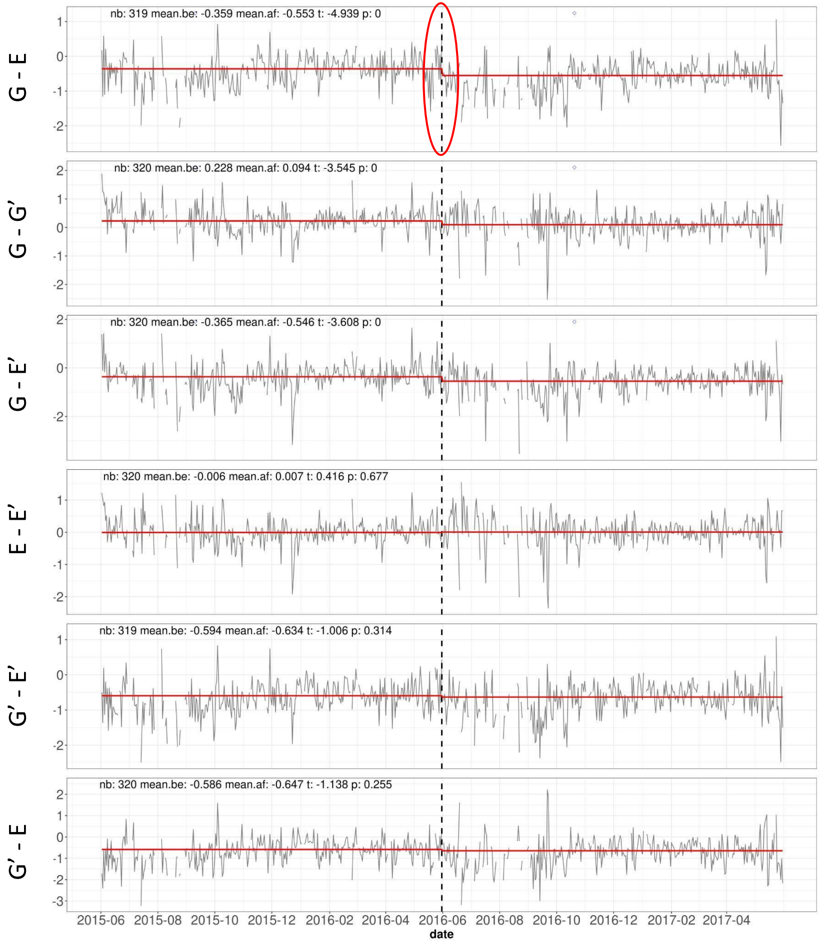
Problem: are the change-points due to GPS or to ERA ?

=> Attribution = procedure used to decide between GPS and ERA



* GNSSseg R package available on the CRAN

Attribution method: test of 6 series of differences



-1

-1

-1

0

0

0

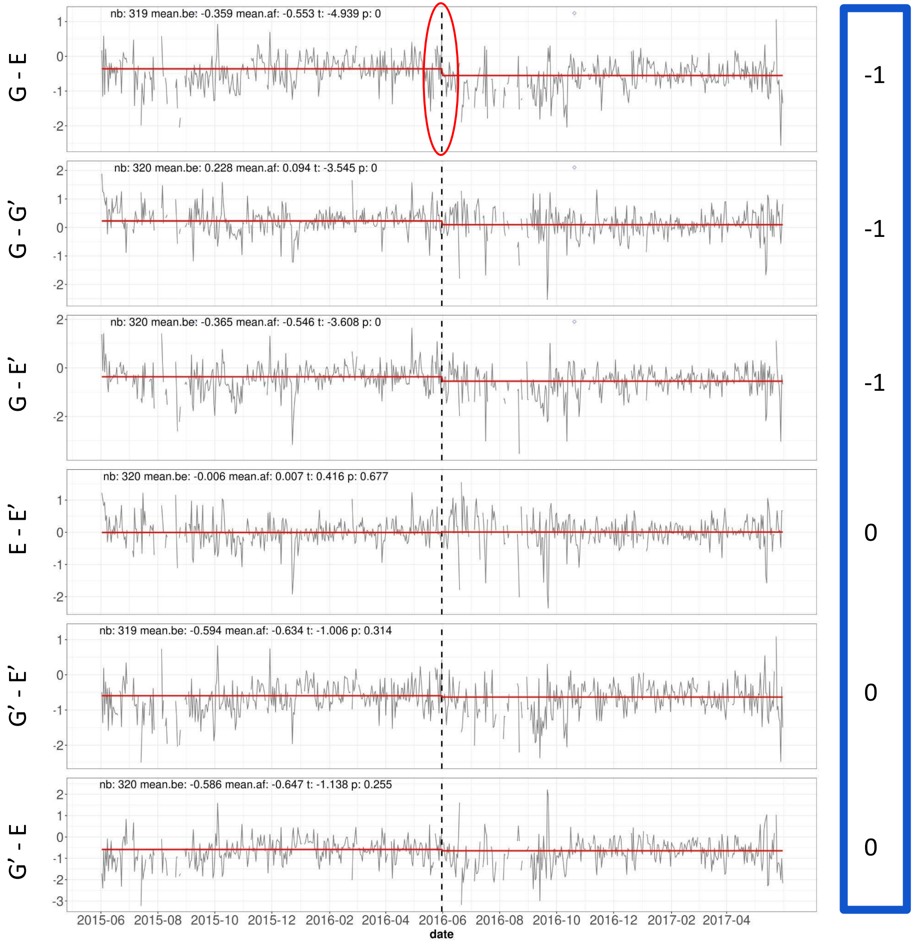
Main station: **G, E**

Nearby station: **G’, E'**

–> 6 differences

1. For each series of difference, estimate offset and perform a significance test

Attribution method: test of 6 series of differences



Main station: **G, E**

Nearby station: **G’, E'**

–> 6 differences

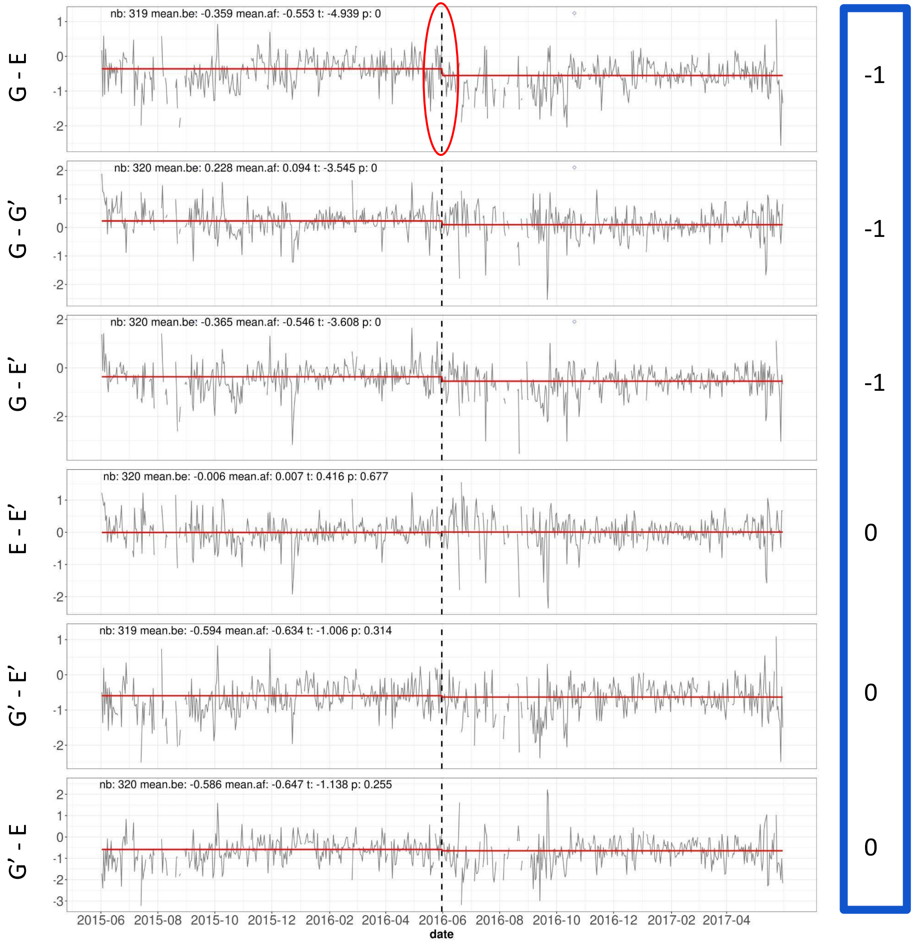
1. For each series of difference, estimate offset and perform a significance test

Test results

Logical table

Truth				Test results					
				1	2	3	4	5	6
G	E	G’	E’	G-E	G-G’	G-E’	E-E’	G’-E’	G’-E
-1	0	0	0	-1	-1	-1	0	0	0
-1	0	0	1	-1	-1	-1	-1	-1	0
-1	0	0	-1	-1	-1	0	1	1	0
-1	0	1	0	-1	-1	-1	0	1	1
-1	0	1	1	-1	-1	-1	-1	0	1
0	1	0	0	-1	0	0	1	0	-1

Attribution method: test of 6 series of differences



Main station: **G, E**

Nearby station: **G', E'**

–> 6 differences

1. For each series of difference, estimate offset and perform a significance test
2. Combine the results from 6 tests to predict in which series the offset occured

Test results

Logical table

Truth				Test results					
				1	2	3	4	5	6
G	E	G'	E'	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
-1	0	0	0	-1	-1	-1	0	0	0
-1	0	0	1	-1	-1	-1	-1	-1	0
-1	0	0	-1	-1	-1	0	1	1	0
-1	0	1	0	-1	-1	-1	0	1	1
-1	0	1	1	-1	-1	-1	-1	0	1
0	1	0	0	-1	0	0	1	0	-1

Prediction

Attribution method: logical table

<u>Two fundamental rules:</u>

(R1) It is unlikely that a change-point in the nearby GNSS series (G') occurs at the same time as a change-point in the main GNSS series (G) or ERA (E)

$$P(G' \neq 0 | G \neq 0 \cup E \neq 0) = 0.1$$

(R2) It is likely that change-points in the reanalysis occur simultaneously with a large spatial extent

$$P(E = E') = 0.9$$

Marginal probability: P(G=0 or E=0) = 0.225 and P(G=1, E=-1) = 0.05, P(G=-1, E=1) = 0.05

	Truth				Conditional probability	Joint probability
	G	E	G'	E'	P(G', E' | G, E)	P(G, E, G', E')
1	1	0	0	0	0,81	0,18225
2	1	0	0	1	0,045	0,010125
3	1	0	0	-1	0,045	0,010125
	1	0	1	0	0,045	0,010125
	1	0	1	1	0,0025	0,0005625
4	1	0	1	-1	0,0025	0,0005625
5	1	0	-1	0	0,045	0,010125
6	1	0	-1	1	0,0025	0,0005625
7	1	0	-1	-1	0,0025	0,0005625
8	0	-1	0	0	0,045	0,010125
9	0	-1	0	1	0,045	0,010125
10	0	-1	0	-1	0,81	0,18225

	Logical table					
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
1	1	1	1	0	0	0
2	1	1	0	-1	-1	0
3	1	1	2	1	1	0
4	1	0	1	0	1	1
5	1	0	0	-1	0	1
6	1	0	2	1	2	1
7	1	2	1	0	-1	-1
8	1	2	0	-1	-2	-1
9	1	2	2	1	0	-1
10	1	0	0	-1	0	1
11	1	0	-1	-2	-1	1
12	1	0	1	0	1	1

	Truncated table					
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
1	1	1	1	0	0	0
2	1	1	0	-1	-1	0
3	1	1	1	1	1	0
	1	0	1	0	1	1
	1	0	0	-1	0	1
4	1	0	1	1	1	1
5	1	1	1	0	-1	-1
6	1	1	0	-1	-1	-1
7	1	1	1	1	0	-1
8	1	0	0	-1	0	1
9	1	0	-1	-1	-1	1
10	1	0	1	0	1	1

Attribution method: outline

For each change-point detected with the segmentation tool (GNSSseg) in the **G-E** series of a main station:

Select a set of **nearby** stations
limit: 200 km (H), 500 m (V)
Each nearby station: **G’ - E’**

→

Test the significance of the jump in **6 series** of differences and **predict** the result for **each pair: main & nearby**

→

Aggregate results from several nearby stations

Segment the **G'-E'** series

Coincident change-points in G'-E' (+/-10 days)

→

Form 6 series of differences (G-E, G-G', G-E'...)

- Limited **time window** where series are **homogeneous**
- Height difference correction
- Outlier screening

→

For each of the 6 series:

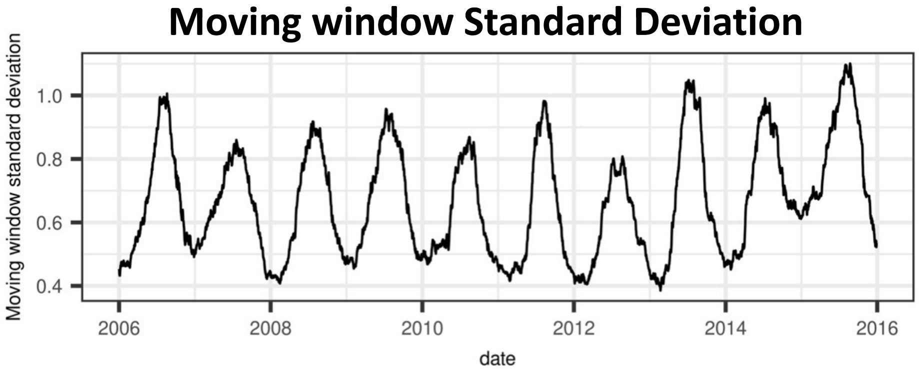
- **Fit a model** including a jump at the position of the change-point
+ Take **heteroscedasticity and autocorrelation** of the series into account
- **Test** if the estimated jump is **significant (level 5%)**

→

Predict the cause(s) of the change-point in the 4 base series (G, E, G', E'):

- **Combination** of 6 test results
- Need a **statistical method** because some test results may be wrong

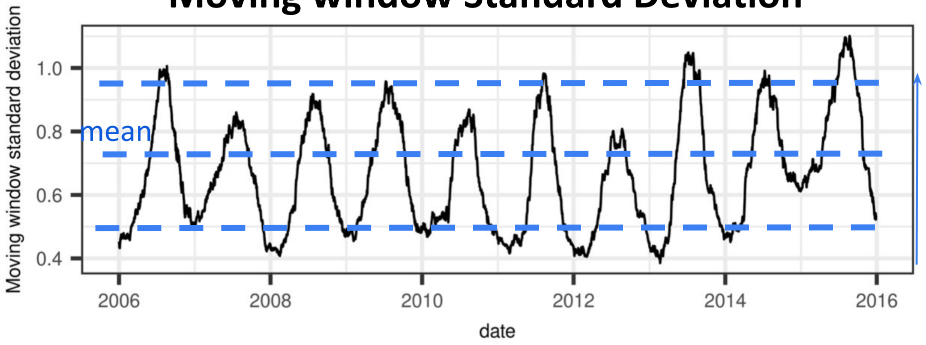
Real data characterization: heteroscedasticity



Window size 60 days

Real data characterization: heteroscedasticity

Moving window Standard Deviation



Window size 60 days

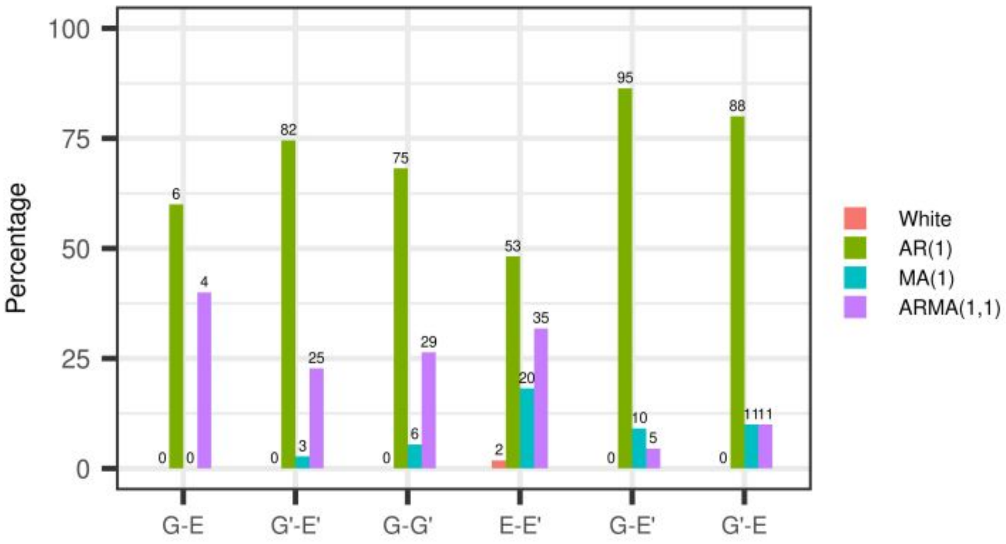
	Mean of MSD		Range of MSD (%)
Distance	< 50 km	> 50 km	
G–E	0.7 ± 0.26		72 ± 20
G’–E’	0.66 ± 0.24		67 ± 19
G–G’	0.52 ± 0.17	1.31 ± 0.47	63 ± 21
E–E’	0.41 ± 0.17	1.26 ± 0.47	73 ± 26
G–E’	0.82 ±0.21	1.38 ± 0.46	67 ± 21
G’–E	0.83 ± 0.26	1.39 ± 0.46	66 ± 20

Full dataset (494 pairs)

- GPS = 55 IGS stations, CODE REPRO2015 solution
- GPS’ = 628 nearby stations, NGL repro3 solution
- ERA, ERA’ = ECMWF reanalysis ERA5

Real data characterization: autocorrelation

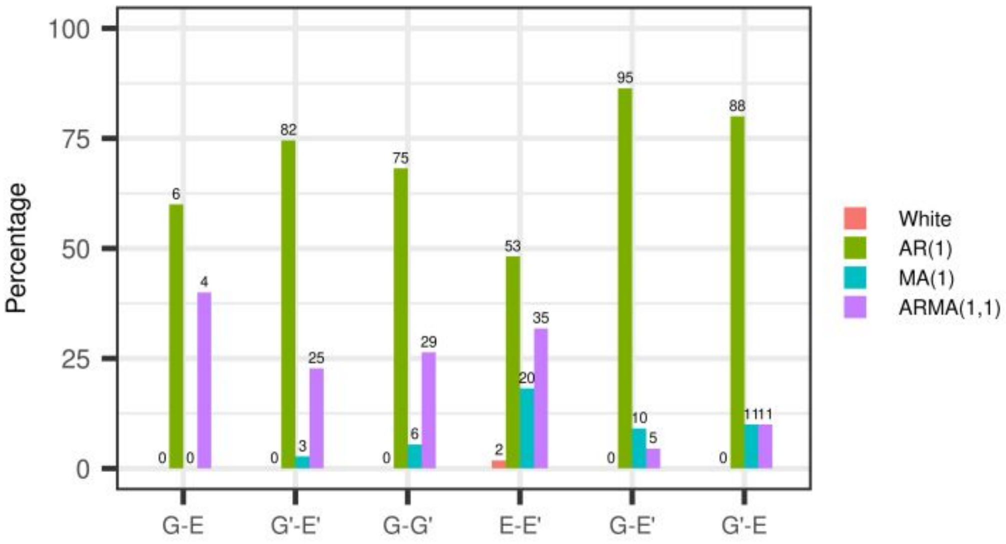
1) Identification of noise model



(R forecast::auto.arima, Hyndman 2008)

Real data characterization: autocorrelation

1) Identification of noise model



(R forecast::auto.arima, Hyndman 2008)

2) Estimation of noise model coefficients

series	AR(1)	MA(1)	ARMA(1,1)	
Coefficients	phi	theta	phi	theta
G-E	0.30	0.00	0.59	-0.33
G’-E’	0.33	0.22	0.61	-0.38
G-G’	0.33	0.19	0.65	-0.31
E-E’	0.31	0.21	0.34	0.23
G-E’	0.33	0.24	0.59	-0.24
G’-E	0.32	0.21	0.57	-0.28

coefficients are estimated by the MLE (*R: arima*)

Regression methods

Regression model

offset

$$z_t = \mu + \delta u_t + f_t + e_t$$

step function $\rightarrow$ $u_t = \begin{cases} 0, & t \leq t^* \\ 1, & t > t^* \end{cases}$ $\quad t = 1, .., n$

seasonal bias $\rightarrow$ $f_t = \sum_{j=1}^{4} a_j sin(\frac{2\pi j}{L}t) + b_i cos(\frac{2\pi j}{L}t)$

error $\rightarrow$ $\boxed{e_t = e_t^* \sigma_t}$

$$e_t^* = \phi e_{t-1}^* + \theta w_{t-1} + w_t$$

Matrix form

$$\mathbf{z} = X\beta + e \qquad e \sim \mathcal{N}(0, \Sigma_0)$$

OLS (ordinary least squares) $\hat{\beta}_{OLS} = (X'X)^{-1}X'\mathbf{z}$

$$Var[\hat{\beta}_{OLS}] = (X'X)^{-1}(X'\Sigma_0 X)(X'X)^{-1}$$

$$\boxed{\widehat{Var}[\hat{\beta}_{OLS}]_{CLM} = \hat{\sigma}_0^2(X'X)^{-1}}$$

variance of coefficients is not correct in case of heteroskedasticity and/or autocorrelation

OLS-HAC (Heteroskedasticity and Autocorrelation Consistent)

$$\widehat{Var}[\hat{\beta}_{OLS}]_{HAC} = (X'X)^{-1}\boxed{\hat{M}}(X'X)^{-1}$$

consistent estimation

GLS (general least squares) $\hat{\beta}_{GLS} = (X'\Sigma_0^{-1}X)^{-1}X'\Sigma_0^{-1}\mathbf{z}$

$$Var[\hat{\beta}_{GLS}] = (X'\Sigma_0^{-1}X)^{-1}$$

This is BLUE when the variance-covariance matrix is known

FGLS (Feasible Generalized Least Squares)

$$\hat{\beta}_{FGLS} = (X'\hat{\Sigma}_n^{-1}X)^{-1}X'\hat{\Sigma}_n^{-1}\mathbf{z}$$

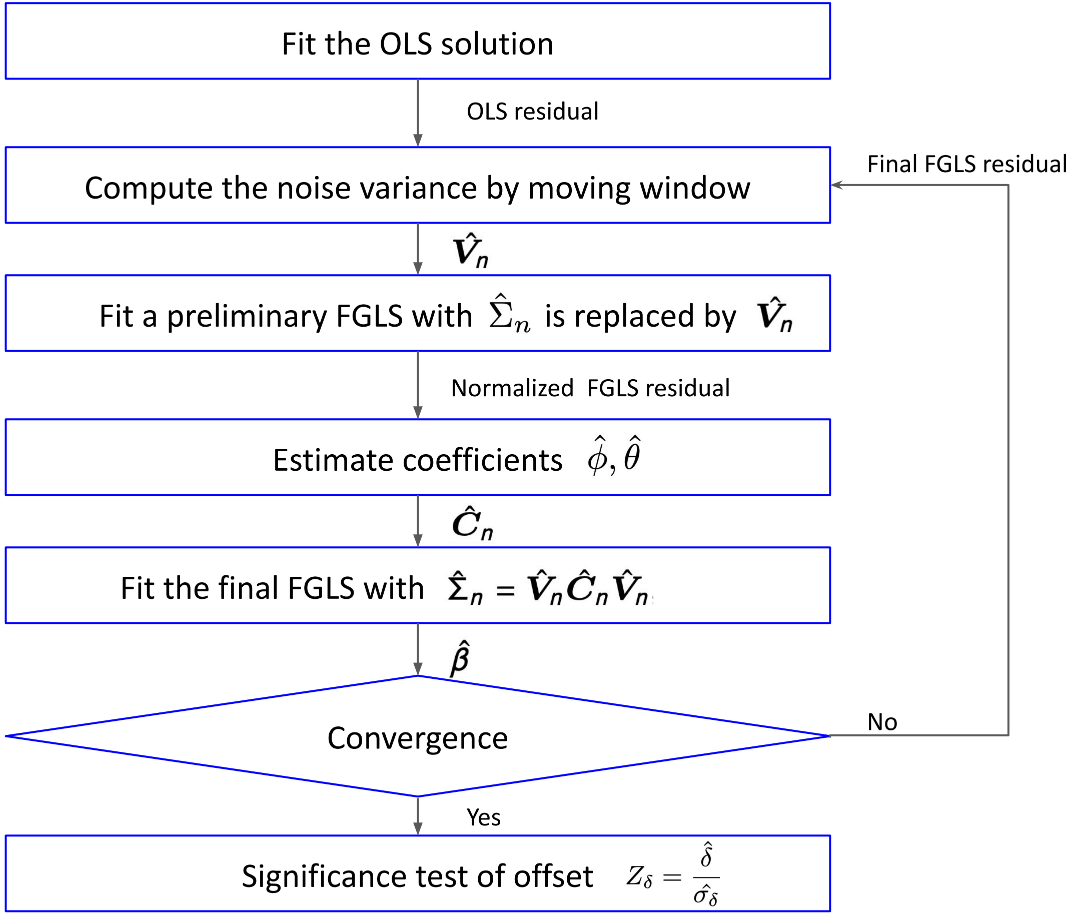
$$\widehat{Var}[\hat{\beta}_{FGLS}] = (X'\hat{\Sigma}_n^{-1}X)^{-1}$$

$\hat{\Sigma}_n = \hat{\boldsymbol{V}}_n \hat{\boldsymbol{C}}_n \hat{\boldsymbol{V}}_n$

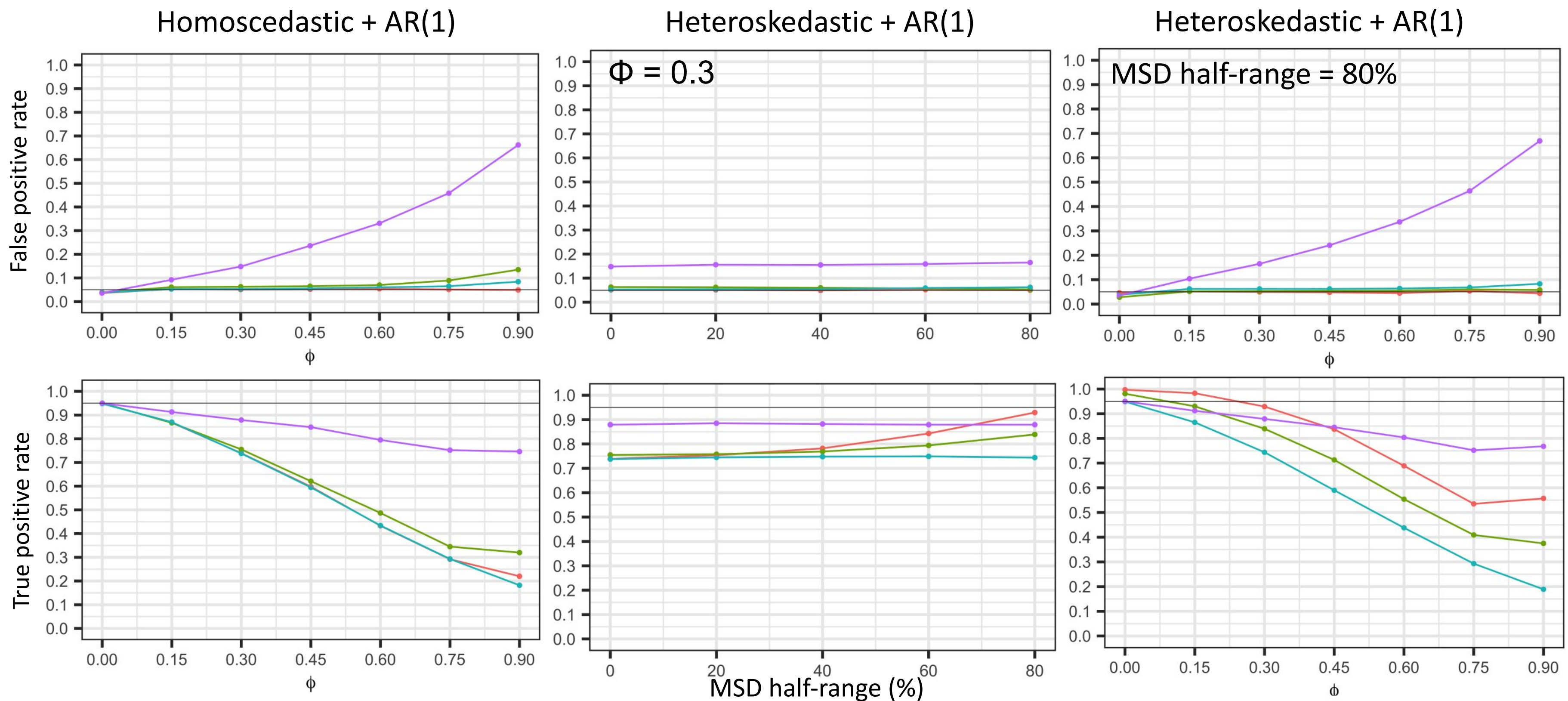
$\hat{\boldsymbol{C}}_n$ correlation matrix

$\hat{\boldsymbol{V}}_n = diag(\hat{\sigma_t^2})$

FGLS regression and test



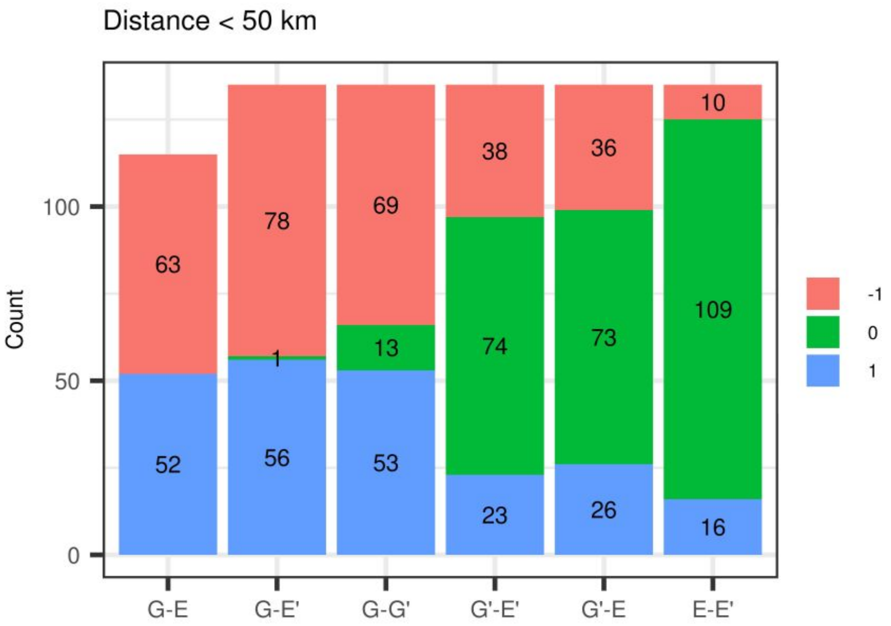
Performance assessment with simulations



TPR = TP/(TP+FN) , FPR = FP/(FP+TN)

FGLS performs best

Test results for real data



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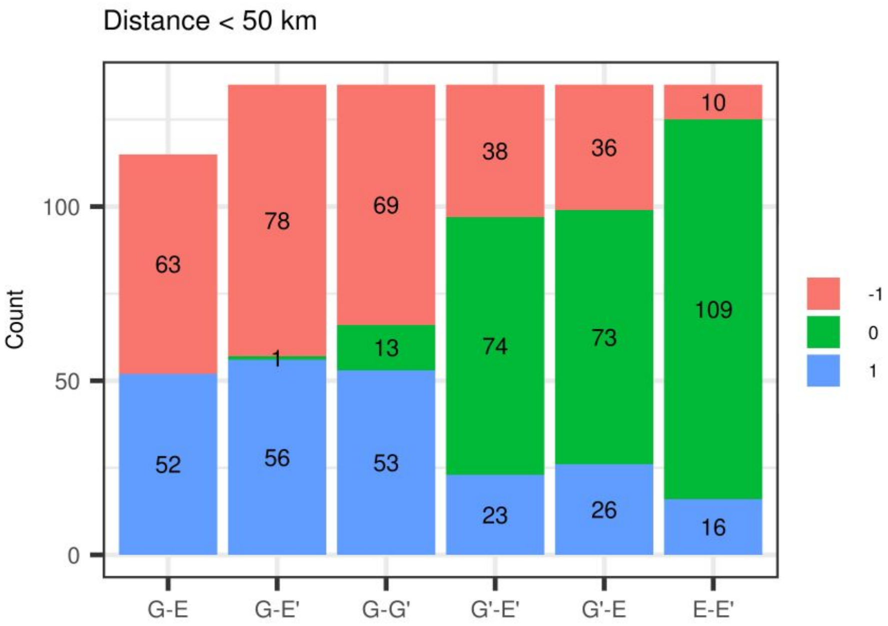
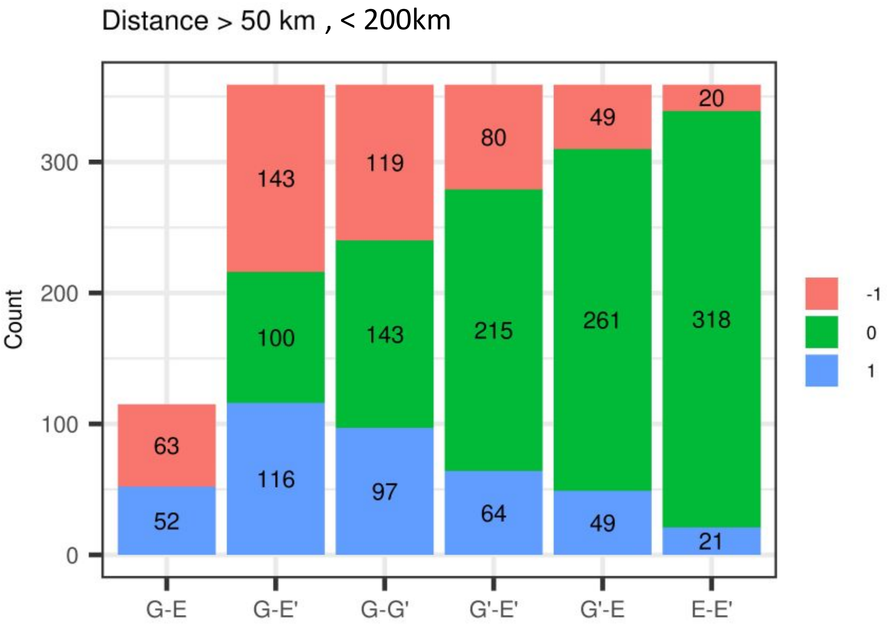
Test results for real data





<u>Full dataset (494 pairs)</u>

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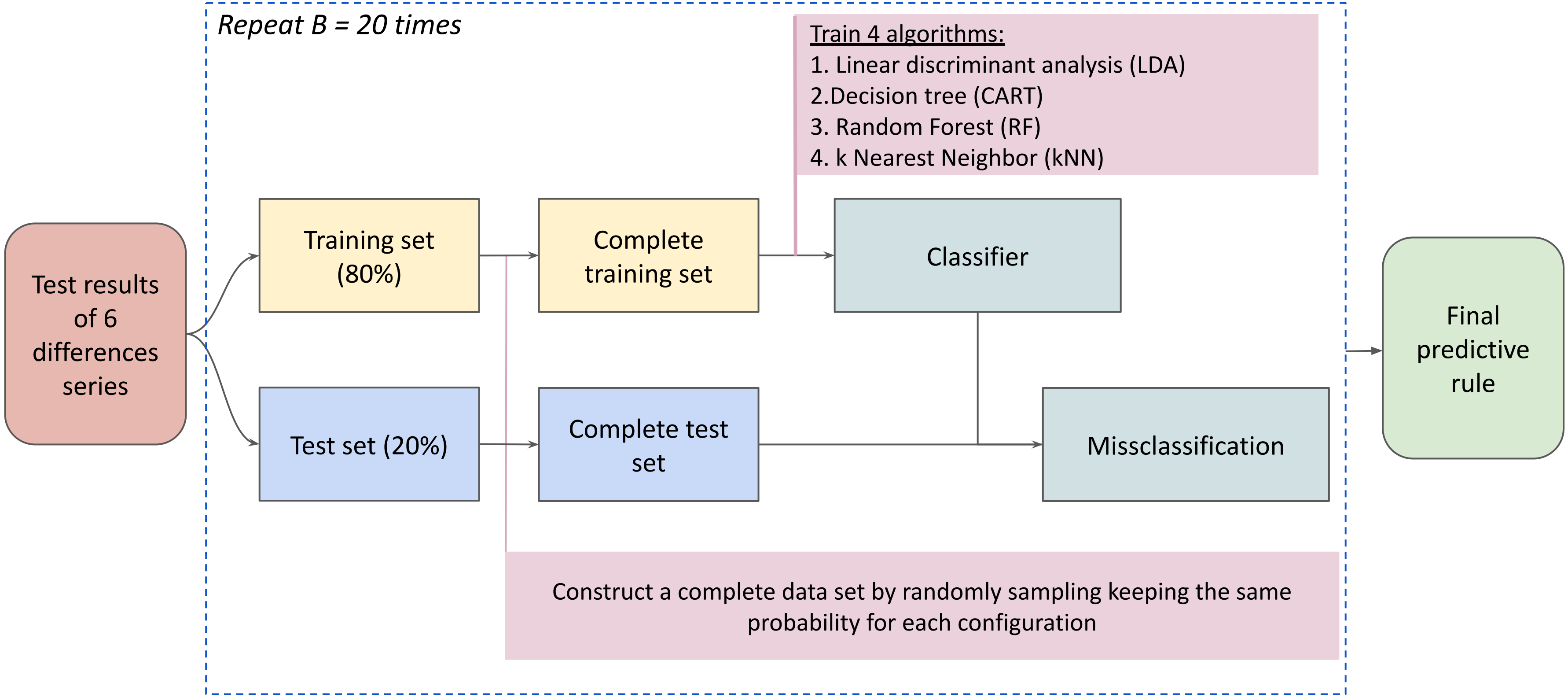
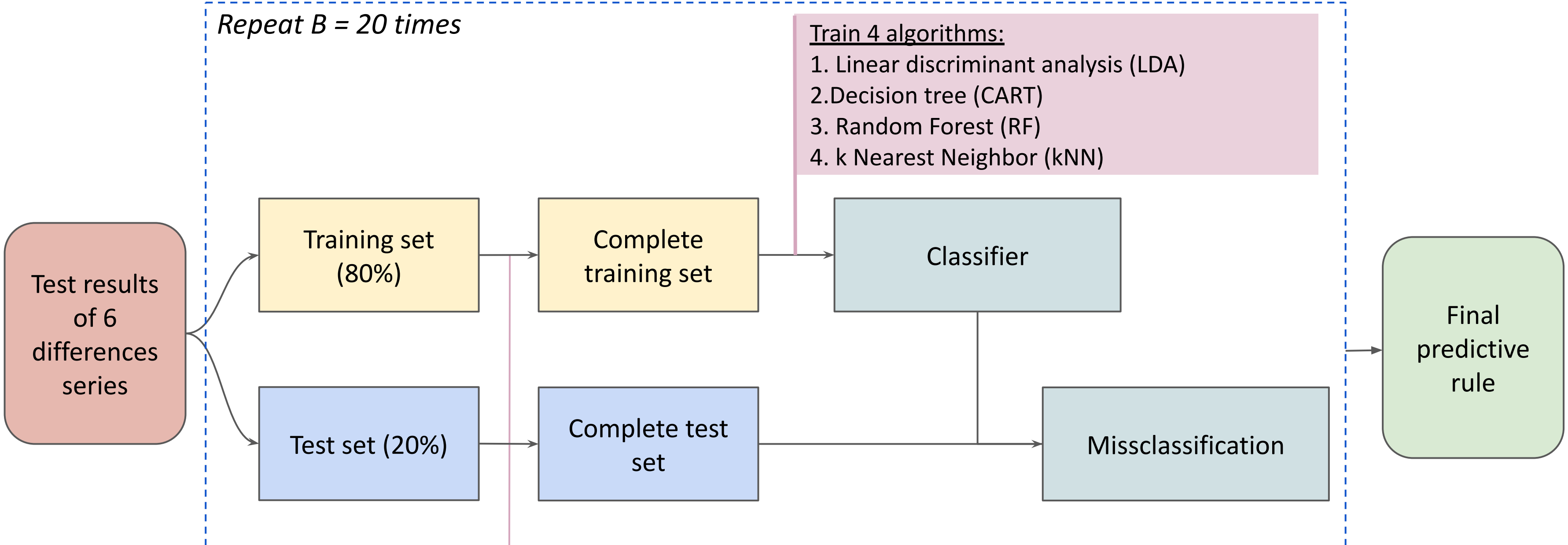
- GPS’ = 628 nearby stations, NGL repro3 solution
- ERA, ERA’ = ECMWF reanalysis ERA5

Predictive rule construction



Predictive rule: cross-validation



c	LDA	CART	KNN	RF
$\overline{err}^c$	0.1463 ± 0.021	0.0142 ± 0.011	0.1412 ± 0.018	0.0049 ± 0.003

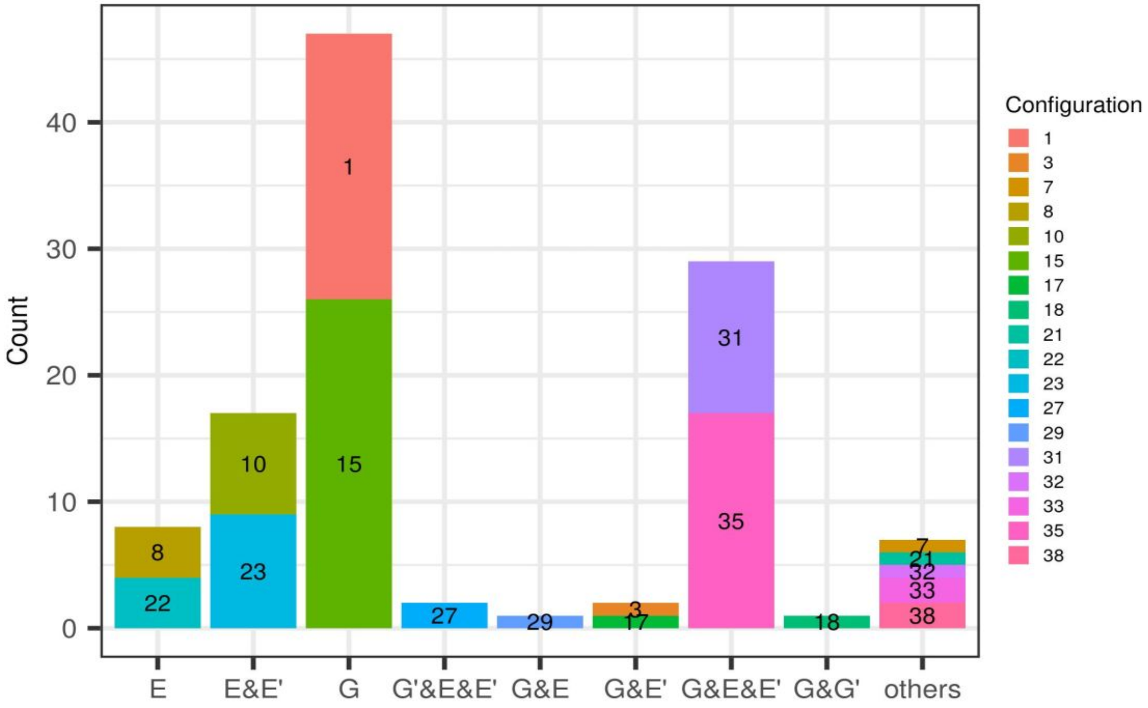
TABLE 3 Mean misclassification error, with the standard deviation in brackets, for the four classifiers.

Prediction results with real data

Significance level: 0.05

Most frequent predicted configurations:

- 1 and 15: offset in GPS (41%)
- 31 and 35: offset in GPS, ERA, and ERA' (25 %)
- 10 and 23: offset in ERA and ERA' (15 %)
- 8 and 22: offset in ERA



Example of test and prediction result

<u>Significance level: 0.05 (CV = 1.96)</u>

Test result:

(-1, -1, -1, 0, -1, 0)

not in the table

Prediction:

configuration 35

(-1, -1, -1, 0, -1, -1)

Break in G,E and E’⟶ low probability

<u>Significance level: 0.01 (CV = 2.58)</u>

Test result:

(-1, -1, -1, 0, 0, 0)

in the table

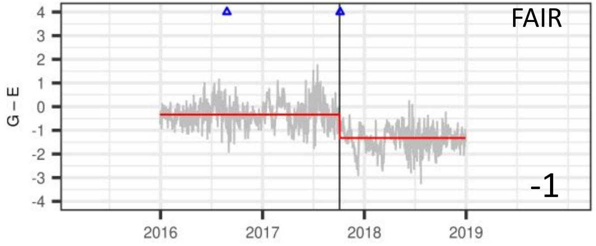
Prediction:

configuration 15

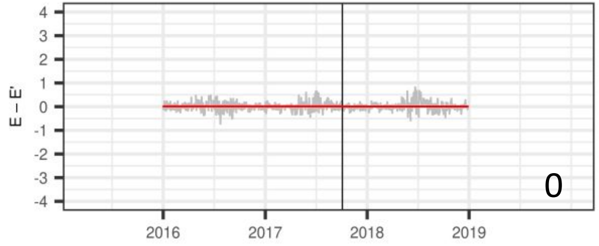
(-1, -1, -1, 0, 0, 0)

Break in G ⟶ high probability

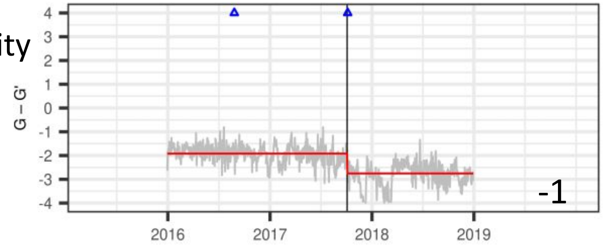
Jump = -0.99, t = -14.6



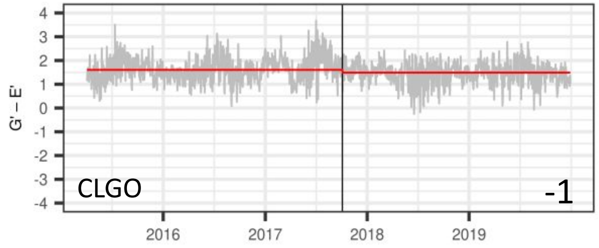
Jump = -0.01, t = -0.67



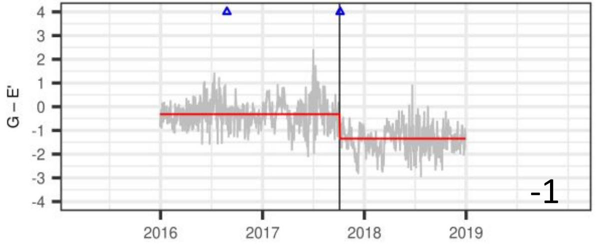
Jump = -0.83, t = -17.2



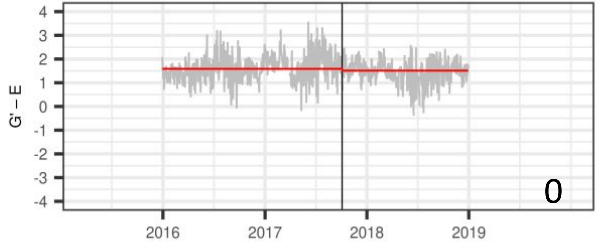
Jump = -0.12, t = -2.43



Jump = -1.03, t = -20.67



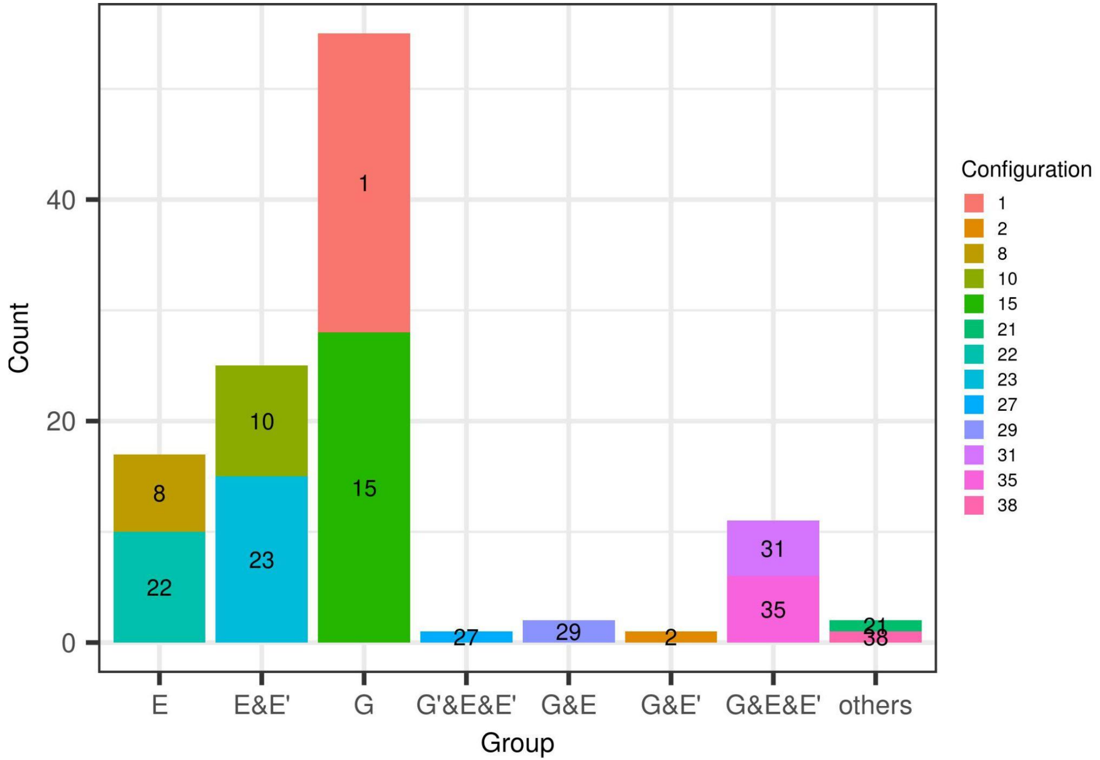
Jump = -0.08, t = -1.71



Prediction results with real data: 1% sig. level

Most frequent predicted configurations:

- 1 and 15: offset in GPS
  41% → 48%
- 31 and 35: offset in GPS, ERA, and ERA'
  25 % →10%
- 10 and 23: offset in ERA and ERA'
  15 % → 22%
- 8 and 22: offset in ERA
  7% → 15%



Conclusion and Perspectives

Conclusions on results:

1. On the regression test:
   - *Heteroskedasticity* and *autocorrelation* are modelled properly with an iterative FGLS procedure
   - *FGLS* works better than the OLS-HAC when the noise has both 2 features

2. On the predictive rule:
   Random Forest outperforms other methods and is able to predict the correct configuration

3. Result on real data
   48% changepoints are attributed to G,
   22% E and E', 10% coincident G,E and E'

Perspectives

1. Improve the attribution:
   - Test the proposed method on a **bigger data set**
   - Improve the robustness and the power of the test procedure by **selecting nearby stations** (distance, length, gaps, etc)
   - Refine the **aggregative rule** (currently based on the distance and number of occurrences of a configuration) when several nearby stations are available

2. Homogenize a more recent (repro3) and denser network to estimate regional and global IWV trends

3. Compare the results to various state-of-the art reanalyses (ERA5, MERRA2, JRA55)

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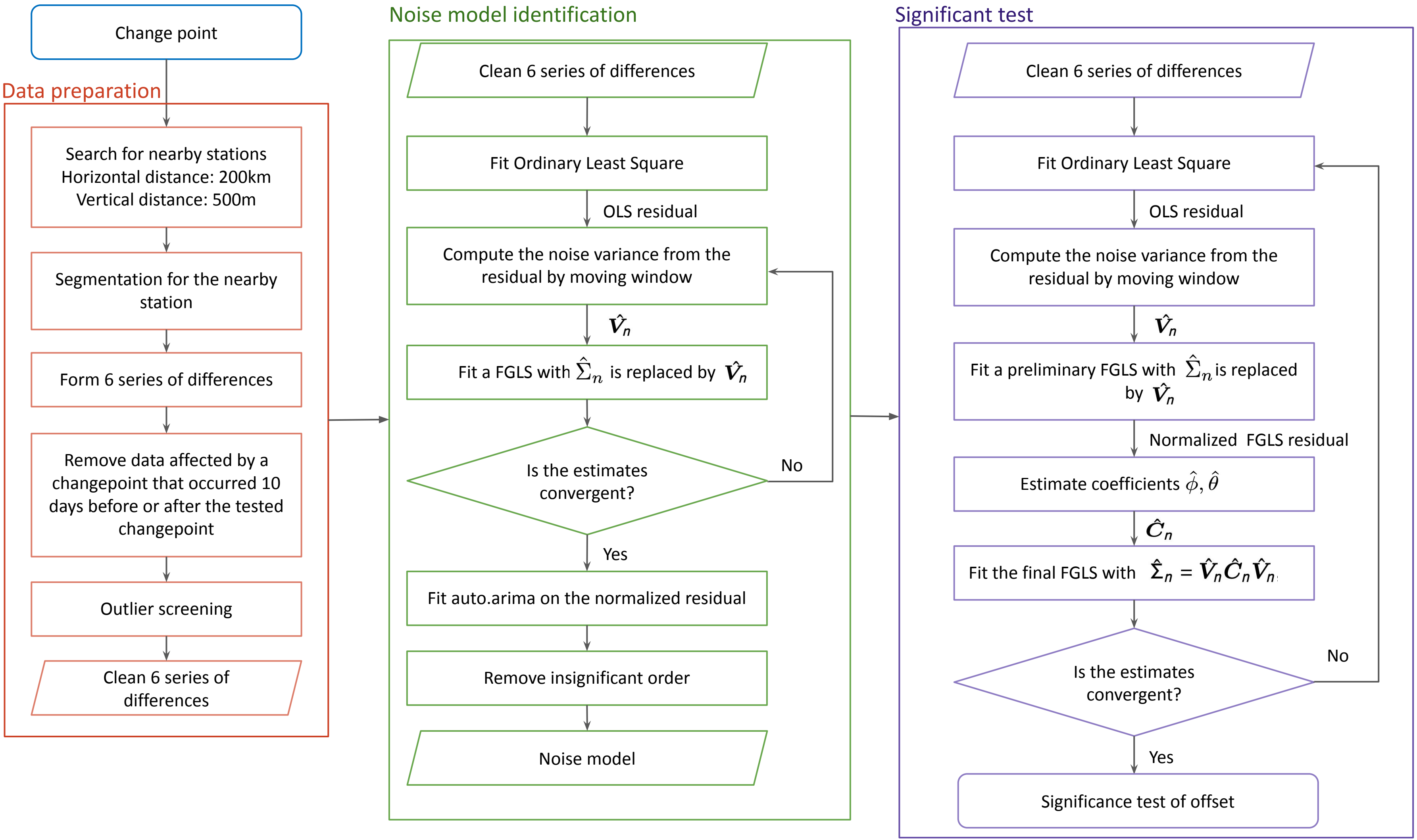
Backup slides


Change point
Data preparation
Search for nearby stations
Horizontal distance: 200km
Vertical distance: 500m
Segmentation for the nearby station
Form 6 series of differences
Remove data affected by a changepoint that occurred 10 days before or after the tested changepoint
Outlier screening
Clean 6 series of differences
Noise model identification
Clean 6 series of differences
Fit Ordinary Least Square
OLS residual
Compute the noise variance from the residual by moving window
$\hat{V}_n$
Fit a FGLS with $\hat{\Sigma}_n$ is replaced by $\hat{V}_n$
Is the estimates convergent?
No
Yes
Fit auto.arima on the normalized residual
Remove insignificant order
Noise model
Significant test
Clean 6 series of differences
Fit Ordinary Least Square
OLS residual
Compute the noise variance from the residual by moving window
$\hat{V}_n$
Fit a preliminary FGLS with $\hat{\Sigma}_n$ is replaced by $\hat{V}_n$
Normalized FGLS residual
Estimate coefficients $\hat{\phi}, \hat{\theta}$
$\hat{C}_n$
Fit the final FGLS with $\hat{\Sigma}_n = \hat{V}_n \hat{C}_n \hat{V}_n$
Is the estimates convergent?
No
Yes
Significance test of offset

Attribution problem

Attribution is a critical problem in the relative homogenization

Pairwise comparison

(+) Utilizes all stations in the network for comparison

(-)When the nearby station has different climatic conditions, it can introduce errors in the differences observed ⟶ difficult to detect inhomogeneities and attribute them correctly.

Composite reference

(+) Reduces inhomogeneities in nearby stations by using a composite reference series.
(+) Automatically attributes changepoints to the main series.
(-) Less efficient when similar inhomogeneities appear in nearby stations or when a large inhomogeneity exists in a single nearby station.

Our study:

- Improved removal of the climatic signal can be achieved by using ERA5 as a reference ⟶ efficient segmentation
- Note that the reference may contain inhomogeneities
- An approach, developped for a sparse network (e.g. single nearby station)

Regression model and test

Regression model

$$z_t = \mu + \delta u_t + f_t + e_t$$

offset

$$u_t = \begin{cases} 0, & t \leq t_k \\ 1, & t > t_k \end{cases}$$

$$t = 1, .., n$$

seasonal bias

$$f_t = \sum_{j=1}^{4} a_j sin(\frac{2\pi j}{L} t) + b_i cos(\frac{2\pi j}{L} t)$$

error $\quad e_t = e_t^* \sigma_t$

$$e_t^* = \phi e_{t-1}^* + \theta w_{t-1} + w_t$$

<u>Matrix formulation:</u>

$$\mathbf{z} = X\beta + \boldsymbol{e}$$

$$\boldsymbol{e} \sim \mathcal{N}(0, \Sigma_0)$$

Regression solution:
–> need to use a Generalized Least Squares (GLS)

Feasible GLS solution

$$\hat{\beta}_{FGLS} = (X'\hat{\Sigma}_n^{-1}X)^{-1}X'\hat{\Sigma}_n^{-1}\mathbf{z}$$

$$\widehat{Var}[\hat{\beta}_{FGLS}] = (X'\hat{\Sigma}_n^{-1}X)^{-1}$$

$$\boldsymbol{\hat{\Sigma}_n} = \boldsymbol{\hat{V}_n \hat{C}_n \hat{V}_n}$$

$\boldsymbol{\hat{C}_n}$ correlation matrix

$\boldsymbol{\hat{V}_n} = diag(\hat{\sigma_t^2})$

Significance test

Null hypothesis $\quad H_0 : \delta = 0$

Test statistic $\quad Z_\delta = \frac{\hat{\delta}}{\hat{\sigma_\delta}}$

Under null hypothesis $\quad Z_\delta \sim \mathcal{N}(0, 1)$

Reject null hypothesis if $\quad |Z_\delta| > 1.96$

Logical table & Truncated table

	Truth				Conditional probability	Joint probability
	G	E	G'	E'	P(G', E' | G, E)	P(G, E, G', E')
1	1	0	0	0	0,81	0,18225
2	1	0	0	1	0,045	0,010125
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	1	0	1	1	0,0025	0,0005625
4	1	0	1	-1	0,0025	0,0005625
5	1	0	-1	0	0,045	0,010125
6	1	0	-1	1	0,0025	0,0005625
7	1	0	-1	-1	0,0025	0,0005625
8	0	-1	0	0	0,045	0,010125
9	0	-1	0	1	0,045	0,010125
10	0	-1	0	-1	0,81	0,18225
11	0	-1	1	0	0,0025	0,0005625
12	0	-1	1	1	0,0025	0,0005625
13	0	-1	1	-1	0,045	0,010125
	0	-1	-1	0	0,0025	0,0005625
14	0	-1	-1	1	0,0025	0,0005625
	0	-1	-1	-1	0,045	0,010125

	Logical table					
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
1	1	1	1	0	0	0
2	1	1	0	-1	-1	0
3	1	1	2	1	1	0
4	1	0	1	0	1	1
5	1	0	0	-1	0	1
6	1	0	2	1	2	1
7	1	2	1	0	-1	-1
8	1	2	0	-1	-2	-1
9	1	2	2	1	0	-1
10	1	0	0	-1	0	1
11	1	0	-1	-2	-1	1
12	1	0	1	0	1	1
13	1	-1	0	-1	1	2
14	1	-1	-1	-2	0	2
15	1	-1	1	0	2	2
16	1	1	0	-1	-1	0
17	1	1	-1	-2	-2	0
18	1	1	1	0	0	0

	Truncated table						
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E	
1	1	1	1	0	0	0	1
2	1	1	0	-1	-1	0	2
3	1	1	1	1	1	0	3
	1	0	1	0	1	1	
	1	0	0	-1	0	1	
4	1	0	1	1	1	1	4
5	1	1	1	0	-1	-1	5
6	1	1	0	-1	-1	-1	6
7	1	1	1	1	0	-1	7
8	1	0	0	-1	0	1	8
9	1	0	-1	-1	-1	1	9
10	1	0	1	0	1	1	10
11	1	-1	0	-1	1	1	11
12	1	-1	-1	-1	0	1	12
13	1	-1	1	0	1	1	13
	1	1	0	-1	-1	0	
14	1	1	-1	-1	-1	0	14
	1	1	1	0	0	0	

Logical table & Truncated table

	Truth				Conditional probability	Joint probability
	G	E	G'	E'	P(G', E' | G, E)	P(G, E, G', E')
15	-1	0	0	0	0,81	0,18225
16	-1	0	0	1	0,045	0,010125
17	-1	0	0	-1	0,045	0,010125
18	-1	0	1	0	0,045	0,010125
19	-1	0	1	1	0,0025	0,0005625
20	-1	0	1	-1	0,0025	0,0005625
	-1	0	-1	0	0,045	0,010125
21	-1	0	-1	1	0,0025	0,0005625
	-1	0	-1	-1	0,0025	0,0005625
22	0	1	0	0	0,045	0,010125
23	0	1	0	1	0,81	0,18225
24	0	1	0	-1	0,045	0,010125
	0	1	1	0	0,0025	0,0005625
	0	1	1	1	0,045	0,010125
25	0	1	1	-1	0,0025	0,0005625
26	0	1	-1	0	0,0025	0,0005625
27	0	1	-1	1	0,045	0,010125
28	0	1	-1	-1	0,0025	0,0005625

	Logical table					
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
19	-1	-1	-1	0	0	0
20	-1	-1	-2	-1	-1	0
21	-1	-1	0	1	1	0
22	-1	-2	-1	0	1	1
23	-1	-2	-2	-1	0	1
24	-1	-2	0	1	2	1
25	-1	0	-1	0	-1	-1
26	-1	0	-2	-1	-2	-1
27	-1	0	0	1	0	-1
28	-1	0	0	1	0	-1
29	-1	0	-1	0	-1	-1
30	-1	0	1	2	1	-1
31	-1	-1	0	1	1	0
32	-1	-1	-1	0	0	0
33	-1	-1	1	2	2	0
34	-1	1	0	1	-1	-2
35	-1	1	-1	0	-2	-2
36	-1	1	1	2	0	-2

	Truncated table						
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E	
15	-1	-1	-1	0	0	0	15
16	-1	-1	-1	-1	-1	0	16
17	-1	-1	0	1	1	0	17
18	-1	-1	-1	0	1	1	18
19	-1	-1	-1	-1	0	1	19
20	-1	-1	0	1	1	1	20
	-1	0	-1	0	-1	-1	
21	-1	0	-1	-1	-1	-1	21
	-1	0	0	1	0	-1	
22	-1	0	0	1	0	-1	22
23	-1	0	-1	0	-1	-1	23
24	-1	0	1	1	1	-1	24
	-1	-1	0	1	1	0	
	-1	-1	-1	0	0	0	
25	-1	-1	1	1	1	0	25
26	-1	1	0	1	-1	-1	26
27	-1	1	-1	0	-1	-1	27
28	-1	1	1	1	0	-1	28

Logical table & Truncated table

	Truth				Conditional probability	Joint probability
	G	E	G'	E'	P(G', E' | G, E)	P(G, E, G', E')
29	1	-1	0	0	0,045	0,00225
30	1	-1	0	1	0,045	0,00225
31	1	-1	0	-1	0,81	0,0405
32	1	-1	1	0	0,0025	0,000125
	1	-1	1	1	0,0025	0,000125
	1	-1	1	-1	0,045	0,00225
33	1	-1	-1	0	0,0025	0,000125
	1	-1	-1	1	0,0025	0,000125
	1	-1	-1	-1	0,045	0,00225
34	-1	1	0	0	0,045	0,00225
35	-1	1	0	1	0,81	0,0405
36	-1	1	0	-1	0,045	0,00225
37	-1	1	1	0	0,0025	0,000125
	-1	1	1	1	0,045	0,00225
	-1	1	1	-1	0,0025	0,000125
38	-1	1	-1	0	0,0025	0,000125
	-1	1	-1	1	0,045	0,00225
	-1	1	-1	-1	0,0025	0,000125

	Logical table					
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E
37	2	1	1	-1	0	1
38	2	1	0	-2	-1	1
39	2	1	2	0	1	1
40	2	0	1	-1	1	2
41	2	0	0	-2	0	2
42	2	0	2	0	2	2
43	2	2	1	-1	-1	0
44	2	2	0	-2	-2	0
45	2	2	2	0	0	0
46	-2	-1	-1	1	0	-1
47	-2	-1	-2	0	-1	-1
48	-2	-1	0	2	1	-1
49	-2	-2	-1	1	1	0
50	-2	-2	-2	0	0	0
51	-2	-2	0	2	2	0
52	-2	0	-1	1	-1	-2
53	-2	0	-2	0	-2	-2
54	-2	0	0	2	0	-2

	Truncated table						
	G-E	G-G'	G-E'	E-E'	G'-E'	G'-E	
29	1	1	1	-1	0	1	29
30	1	1	0	-1	-1	1	30
31	1	1	1	0	1	1	31
32	1	0	1	-1	1	1	32
33	1	0	0	-1	0	1	
34	1	0	1	0	1	1	
35	1	1	1	-1	-1	0	33
36	1	1	0	-1	-1	0	
37	1	1	1	0	0	0	
38	-1	-1	-1	1	0	-1	34
39	-1	-1	-1	0	-1	-1	35
40	-1	-1	0	1	1	-1	36
41	-1	-1	-1	1	1	0	37
42	-1	-1	-1	0	0	0	
43	-1	-1	0	1	1	0	
44	-1	0	-1	1	-1	-1	38
45	-1	0	-1	0	-1	-1	
46	-1	0	0	1	0	-1	