Internal waves and internal boundary currents
In memoriam Fridtjof Nansen

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Abstract—Re-reading of books written by Fridtjof Nansen after his great travels and humanitarian feats shows that his precise observations as well as deep thoughts are still rich source of inspiration. He gave clear description of the behavior of this ship Fram in dead waters of Siberian seashores and invited Vagn Ekman to clarify the phenomenon in laboratory experiments. Laboratory studies of stratified flows is now performed in many laboratories around the globe. The first part of this paper is a memorial talk of Nansen. In the second part a classification of infinitesimal periodic motions including waves and singular components of motions is presented. Singular solutions describe a set of linear periodic boundary layers. Approximation of a 3D homogeneous fluid results in merging boundary layers and degeneration of the governing equations set. Structure of 2D attached internal waves is visualized numerically.

Key-words: viscous stratified fluids, exact regular singular solutions, periodic motions

1. Introduction — life in the service of science and humanity

Fridtjof Nansen was born in Norway, on October 10, 1861 in aristocratic family. His father, Baldur Fridtjof Nansen, a prosperous lawyer, was a religious man with a clear conception of personal duty and moral principle. His mother, baroness Adelaide Johanna Tekle Isidore Belling formerly Wedel Jarlsberg was a strong-minded, athletic woman. She owned a farmstead near Christiania (now Oslo) where Fridtjof Nansen, together with his brother Alexander and a number of older half-brothers and half-sisters, had a privileged childhood. Mother had introduced children to outdoor life and encouraged them to develop physical skills. Not massively built, F. Nansen was tall, supple, strong, and possessed the physical endurance to ski fifty kilometers in a day and then to win a skiing competition.
Preferring physics and mathematics he, nevertheless, selected zoology to spend more time in the open air, when in 1880 he entered the University of Christiania. Even during student days, in 1882, when, at a tutor’s suggestion, he took passage aboard a sealer Viking to the Arctic Ocean, he used thermometers, bathometers, and different nets for scientific observations of winds, ocean currents, ice movements, and animal life. He firstly found that warm salty Atlantic waters were placed below cold and fresh Arctic waters and rightly described freezing of polar seas. He landed on a floating iceberg and collected stones. Thinking about the patterns of the frozen in ice ship drift, he recognized that the effect of complex vortex currents caused by the bottom topography dominate, over a wind impact. He wrote extended diaries and published papers about his expeditions in newspapers and scientific journals that were illustrated by his own excellent sketches. After the travel he firmly decided to study cold Arctic Ocean and crossed Greenland icecap on skies.

On his return Nansen was offered the post of curator at the Bergen Museum. During six years he performed intensive laboratory study mostly with the microscope that was a gift of his father. The transition from the rugged days aboard the Arctic sealer to study minute animals through microscopes was productive. He wrote his thesis *The Structure and Combination of Histological Elements of the Central Nervous System* in English language in 1887, accepted by the examining board with a degree of skepticism. He could obtain his doctorate only four days before leaving for Greenland in view of his future dangerous travel. In the course of two years he published abbreviated version of his thesis in four different languages and today his thesis is regarded as a classic. He visited the international biological station in Naples (Italy) in 1886 and was so impressed that recommended to construct similar stations in Norway. The idea was supported and two biological stations were built on Nansen’s projects. Constructions were so good that the station in Drobak is still operating today, in the same building.

In 1887 Nansen embarked on the preparations for the journey to cross the Greenland icecap. The six-man expedition was financed by a wealthy Danish businessman A. Gamel and Nansen himself. The expedition started in June on the east coast and going to the inhabited west coast. The ice drift delays the beginning of the journey as it took 12 days to row to land in open boats instead of expecting 2–3 hours. Nansen, still only 27 years of age, had led his team firmly and finally reached the west coast in late September. Throughout the journey the team made careful records of meteorological conditions when the temperature fell to 50 ºC below zero. Data were analyzed later by the known meteorologist H. Mohn (1835–1916). No boats were due to leave Greenland until the following spring, so Nansen spent the enforced winter in Greenland studying the Inuits and returned to Norway in May 1889. After the expedition he published many articles and books, like *The First Crossing of Greenland* (1890) and *Eskimo Life* (1891).
Meanwhile, he started planning an another great expedition by ship over the Polar Sea hoping to reach the North Pole. He read a lot of lectures in different countries discussing his plans and construction of the ship. When – in 1892 – he outlined his plan for a North Pole expedition in the Royal Geographic Society of London, he met a lot of criticism and opposition by the most experienced Arctic explorers. But the idea received a great support in Norway. The Storting (the Norwegian Parliament) granted a large part of the necessary expenses, subscriptions from the King, private individuals and own payment of Nansen provided the rest.

The primary task was to build a ship that could withstand the pressure of the ice. Nansen collaborated with famous shipbuilder Colin Archer to design it. The government provided by the best oak and timber for the ship construction from the state stock. The hull of the vessel Fram was made exceptionally strong, and her lines below the waterline were far rounder than customary. As a result, when the pack became jammed hard together, the ship slipped free and was lifted clear instead of being crushed.

In June 1893 the Fram left Christiania with provisions for six years and fuel oil for eight. Nansen believed the trip would take from two to three years. After having sailed the northern Siberian coast, the expedition reached the area around New Siberian Islands where the Fram had been frozen fast in the drifting ice. However, it became evident soon, that the ship root was too far from the North Pole. Then Nansen with Hjalmar Johansen left the Fram on February 1895 to ski to the North Pole with dog sledges. Despite the incredible difficulties they reached 86°14’N when the drifting ice and lack of food forced them to turn back. The travelers had no idea of Fram’s whereabouts, so they decided to spend the winter in Franz Josef Land, and they survived in a stone hut by shooting walruses and polar bears. After the long winter they started to go to south, and by an incredible stroke of luck, they met a British expedition, headed by Frederick George Jackson, which took them back to Norway. Just after Nansen had arrived in Norway, the Fram came in, too. Nansen described the history of the expedition in a two-volume work Farthest North (1897). The successful outcome of the North Pole expedition made Nansen a national hero and gave him a world-wide reputation.

The most important results of Nansen’s Fram expedition were

- the discovery of the deep Arctic Ocean completely devoid of islands,
- the confirmation of the existence of the trans-polar current,
- existence of intermediate warm and salty waters,
- thickening pack ice from below due to freezing of melting fresh water.

Furthermore, Nansen recognized that the Fram and the ice pack drifted approximately 30° to the right of the wind direction. This fact he interpreted as the effect of the Earth’s rotation, which laid the concept for the Ekman spiral and the foundation for the modern wind-driven ocean circulation. Nansen described in details the misterious “dead water” phenomen which the Fram met
in Bergen fjord and Siberian coast waters. Scientific results of expedition were published in a six volumes collection of papers. Later Nansen decided to hand the *Fram* over to Roald Amundsen for an expedition to the South Pole. Nansen and Captain Otto Sverdrup shared their rich experience on ship motion in ice with the Russian Admiral S.O. Makarov, who decided to built the first large icebreaker with iron hull.

As a scientist, Nansen was much aware of the need for precise and exact measurements. Leaving the ship in March 14, 1895 he wrote in a letter to Sverdrup: “Besides foods, weapons, clothes, and equipment, take off scientific materials, diaries, and collections, but not so heavy, ...photos and plates... It is good to take off Oderman’s densimeter for measuring of sea water density…”. Later Nansen found that some of the oceanographic measurements on the *Fram* were not of sufficient precision. He wrote: “I understood that future studies in physical oceanography would have small impact or even no sense if they will not produced by more exact methods than current methods or used before”. At that time, when densimeters were the most precise instruments in physical oceanography, Nansen exchanged his experience of methods of calibrations with the known Russian oceanographer Admiral S.O. Makarov (1848–1904), who made round-the-world expedition on the corvette *Vityaz* (1886–1889, 903 days of travel and 526 days of sailing and steaming) and wrote the famous book *Vityaz and the Pacific Ocean*. Nansen was able to remarkably improve methods and instruments, and he invented the bottle for sampling ocean water at various depths and different types of current meters. He supported the international cooperation in oceanography, and he was one of the founding fathers of the International Council for the Exploration of the Sea (ICES) in 1902.

Nansen’s books were soon translated into Russian (first in 1897) making his name extremely popular. He was honored by the Prince Konstantin Gold Medal of the Russian Geographic Society in 1897. He visited Russia first in 1898, when he was awarded by the State Stanislav insignia and was elected Honor Member of St. Petersberg Academy of Science.

The success of the *Fram* expedition stimulated Norwegians to act for state independence, and Nansen involved himself in the political debates. He played an important role in the actions in 1905 when the union with Sweden was dissolved and Norway declared its full independence. Reportedly, he was also secretly requested to become either president or king but declined both offers, on the grounds that he was “a scientist and explorer”. However, he played a personal part in bringing to the vacant Norwegian throne the Danish Prince Carl, who took the Norwegian name Haakon VII and was appointed as Norway’s first ambassador in London.

After two years in London, he returned to scientific work for some years and had some good years studying the oceanography of the Norwegian Sea and the way of formation of bottom water in the Greenland Sea. His results were published jointly with Professor B. Helland-Hansen in the classic book *The Norwegian Sea*. 
In 1913 he traveled to Siberia and Far East with a diplomat I.G. Loris-Melikov and the deputy of the State Duma (Parliament), S.V. Vostrotin from Tromsoe to Krasnoyarsk. They followed the only working way from Far East to Europe up to the still mouth of the river Enisey on a cargo ship _Correct_, then up along the river Enisey on a motor boat to Krasnoyarsk. Then he took a rail car and went to Vladivostok on the just constructed Transsiberian railway, where one of the stations was named by him. The Eastern Siberian railroad is the longest continuous railway of the world. His impressions of the environment, common people life, and seeing of future development of these extended but almost unhabited lands are expressed in his excellent book _Into the country of Future (Fremtidens Land) – Great North Marine way from Europe to Asia through Kara Sea_ with a supplement _Shipping in Kara Sea_. The book contains right prognosis of future economic development of this outermost land and was reprinted in Russia (_Nansen_, 1992).

The further life of Fridtjof Nansen is an illustration of his deep thinking, efficiency, and humanism. He did save the lives of millions of people. In 1917–1918 Nansen negotiated in Washington, USA an agreement for a relaxation of the blockade of the allies to permit shipments of essential food to neutral Norway. In June 1921 the Council of the League of Nations, spurred by the International Red Cross and other organizations, instituted its High Commission for Refugees and asked Nansen to administer it. For the stateless refugees under his care Nansen invented the “Nansen Passport”, a document of identification, which was eventually recognized by fifty-two governments. In the nine-year life of this office, Nansen ministered to hundreds of thousands of refugees – Russian, Turkish, Armenian, Assyrian, Assyro-Chaldean – utilizing the methods that were to become classic.

A heavy drought in the Russian grain growing areas in 1921 brought famine to millions of people. Nansen responded on appeal of Maxim Gorky and opened in Moscow Kremlin an office of the International Russian Relief Executive. But his appeals to the League of Nations for funds to finance the work met deaf ears. As Nansen was sure that even a small food parcel can save a life, he appealed to common people and succeeded in raising finances. Norwegian pensioners, peasants, and charity organizations collected 3,225,295 Norwegian crowns, and the government added 770,000 crowns to this sum. Nansen was proud that the small populated Norway gave more for struggle with famine in Russia than great states. Although this amount was not sufficient to save all of the starving people and many of them died, thousands received help and survived, particularly in Ukraine and the Volga districts (the figures quoted are ranging from 7 to 22 millions). Nansen was made an honorary member of the Moscow Soviet and honored by Thanking Letter of the Russian Supreme Soviet. Together with Maxim Gorky he wrote the book _Russia and Peace_ (Russland og freden), where he defended Lenin’s harsh methods and argued that they were necessary in order to build up the country.
In recognition of his work for refugees and the famine-stricken, the Nobel Committee in Oslo decided to honor Fridtjof Nansen with the 1922 Nobel Prize for Peace. A Danish publisher, Christian Erichsen, presented him with the same sum. Half of the total amount Nansen spent to create two agricultural stations: one in the Saratov region and the second in Ukraine. Idea of machinery stations and rural agricultural institutes was reborn in the 1930’s, when kolkhozes replaced the individual rural economy.

Nansen arranged an exchange of about 1,250,000 Greeks living on Turkish land for about 500,000 Turks living in Greece, with appropriate indemnification and provisions. Nansen’s fifth great humanitarian effort was to save the remnants of the Armenian people from extinction. He drew up a political and financial plan for creating a national home for the Armenians in Erivan, which was not accepted by the League of Nations. By personal efforts he settled later some 10,000 Armenians in Erivan and 40,000 in Syria and Lebanon. Impressed by Armenian history and fortune, Nansen wrote the books *Armenia and the Middle East* (1927), *Along Armenia* (1929), and *Across Caucas to Volga*, and the made money-raising tours in USA for the Armenian people.

During all these years Nansen remained interested in Arctic studies. He initiated the creation of the Aeroarctic Society, was elected as its Permanent President in 1924, and took part in the Second Aeroarctic Congress in St. Petersburg in 1928. In 1930 he planned to take part in an Arctic expedition on the dirigible balloon *Count Zeppelin* when he died from a heart attack on May 13, 1930. Nansen was buried on May 17, on the national celebration day of Norway.

The world lost a great humanist, scientist, traveller, and writer. The memory of his humanitarian actions is still alive in different countries and especially in Russia in the hearts of the descendants of millions of saved people.

2. Discovery of internal waves

Although mathematical studies of waves on an interface between fluids of different densities was initiated by Stokes (1847), physical significance of the phenomenon as well as their existence inside a continuously stratified fluid were not recognized up to the *Fram* expedition. The pioneering paper of Rayleigh (1880), containing the definition of buoyancy frequency was not properly written up to the late twentieth of the next century, when Väisälä (1925) and Brunt (1927) rediscovered this principle parameter for a stratified atmosphere.

Nansen gave a picturesque description of the “dead water” (Ekman, 1906) and later interpreted the semidiurnal variation of the ocean temperature as the manifestation of internal waves. The deepness of his understanding is illustrated by a letter to the Russian polar explorer Barony E.V. Tall, in which he recommended to register the lost of the ship velocity, to measure thickness of upper fresh layer, and to measure the difference in salinities below the the ship and under the boat which is towed on different distances from the ship.
(Pasetskii, 1986). He discussed the problem with V. Bjerknes, who asked his former student to model the “dead water” phenomenon in a laboratory. Nansen provided money for organizing the experiment, Bjerknes found a grant for the work, Ekman performed a comprehensive study of forces acting on the model of Fram and visualized the patterns of waves on the interface. Unfortunately, the resulting paper (Ekman, 1906) did not receive continuation up to the late fifties of the last century, when Lighthill (1978) started to study anisotropic waves and collected theoreticians and experimenters for cooperative studies on the internal waves in a continuously stratified liquid. Results were also applied for qualitative explanation of mysterious phenomena like “turbulence of the clean sky” caused by overturning of internal waves, “mountain lee waves” in the atmosphere and waves behind moving vehicles in the ocean. The number of work increased greatly and setups for internal waves modeling were created in different countries. As the results of the studies based of different approaches were in general in agreement while not fitting completely, the classification of infinitesimal periodic motions was done (Chashechkin and Kistovich, 2004) and its results were used for construction of exact solutions of some internal wave generation problems (Chashechkin et al., 2004; Bardakov and Chashechkin, 2004).

3. Classification of infinitesimal periodic motions

Periodic motions are studied on rotating with angular velocity $\Omega$ spherical planet with the gravity field characterized by gravity acceleration $g$. The density profile $\rho(z) = \rho_o \exp(-z/\Lambda)$ is defined by the total salinity $S = \Sigma S_n$ that is of concentration of dissolved or dispersed matter $S_n$ and is characterized by a scale $\Lambda = (d \ln \rho(z)/dz)^{-1}$ and frequency $N = \sqrt{g/\Lambda}$ of buoyancy, which can be supposed to be constant and by $N_c = \sqrt{(N^2 c_s^2 - g^2)/c_s^2}$ where $c_s$ is sound velocity. Disturbances of density $\rho = \rho \left(p, S_n, |v|\right)$ depend on velocity $v = (u, v, w)$ pressure $p$, and $S_n$.

The set of governing equations in the linear approximation has the form

$$\frac{\partial \rho}{\partial t} - \frac{w}{\Lambda} + \nabla \cdot v = 0, \quad \frac{\partial S}{\partial t} - \frac{w}{\Lambda} - \kappa \Delta S = 0$$

$$\frac{1}{c_s^2} \frac{\partial \rho}{\partial t} - \frac{w g}{c_s^2} + \nabla \cdot v + \kappa \Delta S = 0$$

$$\frac{\partial u}{\partial t} = -\frac{\partial \rho}{\partial x} + 2\Omega \left(v \sin \varphi - \frac{1}{\sqrt{2}} w \cos \varphi\right) + v \Delta u + \left(\mu + \frac{v}{3}\right) \frac{\partial}{\partial x} \nabla \cdot v$$

$$\frac{\partial v}{\partial t} = -\frac{\partial \rho}{\partial y} + 2\Omega \left(\frac{1}{\sqrt{2}} w \cos \varphi - u \sin \varphi\right) + v \Delta v + \left(\mu + \frac{v}{3}\right) \frac{\partial}{\partial y} \nabla \cdot v$$

$$\frac{\partial w}{\partial t} = -\frac{\partial \rho}{\partial z} + \sqrt{2} \Omega (u) \cos \varphi + v \Delta w + \left(\mu + \frac{v}{3}\right) \frac{\partial}{\partial z} \nabla \cdot v - \rho g$$
where $\bar{\rho}$, $\bar{\rho}$, $\bar{S}$ are the pressure minus hydrostatic pressure, medium-density perturbation normalized to the density at the reference level $z=0$, and normalized on salinity profile perturbation, $\varphi$ is the latitude of the observation point, and $\nu$ and $\mu$ are the first and second kinematic viscosities.

In particular cases the set (1) had to be supplemented by initial and no-slip and no-flux boundary conditions $\mathbf{v} = \mathbf{I}_n \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is local normal to the contact solid boundary $\Sigma$. In the present study the scale of stratification is large and dissipative coefficients of kinematic viscosity $\nu$ and salt diffusion $\kappa_s$ are small. Axis $z$ of the Cartesian coordinate frame $(x, y, z)$ is directed to zenith and the $x$- and $y$-axes are taken so that the corresponding projections of the angular velocity are equal to each other.

For periodic flows $\mathbf{v} = \mathbf{v}_0 f_p(\mathbf{k}, \omega)$, $P = P_0 f_p(\mathbf{k}, \omega)$, $\rho = \rho_0 f_p(\mathbf{k}, \omega)$ where $f_p(\mathbf{k}, \omega) = \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$ with real positive frequency $\omega$ and wave vector $\mathbf{k} = (k_x, k_y, k_z)$, the general solution of to system Eq. (1) can be written as a superposition of elementary waves

$$A = \sum_{j} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} a_j(k_x, k_y) \exp \left( i \left( k_{zj} (k_x, k_y) z + k_x x + k_y y - \omega t \right) \right) dk_x dk_y,$$

where $A$ is a velocity component, pressure, or density. The summation must be performed over all roots of the dispersion equation that are obtained from the condition of non-trivial solvability of the equation set Eq. (1). Coefficients $a_j(k_x, k_y)$ are defined from the boundary conditions. For stationary periodic waves, frequency $\omega$ is fixed, and the dispersion equation describes the relation between the wave number components that is expressed for given $k_x$ and $k_y$ values and has a form

$$D_{\kappa} \left\{ \omega D_{\nu}(k) \left[ \omega D_{\nu}(k) \bar{D}_{\nu}(k) + 2\sqrt{2} \omega \Omega \left( k_x - k_y \right) \cos \varphi \right] \right.$$

$$\left. - \omega D_{\nu}(k) N^2 \left[ D_{\nu}(k) + i \left( \mu + \nu/3 \right) k_\perp^2 \right] + \right.$$

$$\left. + 4\omega \Omega^2 \left[ N^2 \sin^2 \varphi - \omega \left( D_{\nu}(k) + i \left( \mu + \nu/3 \right) f^2(k) \right) \right] \right.$$

$$\left. + \frac{\kappa^2 k^2}{\Lambda} \times \left[ \omega k_z D_{\nu}^2(k) - 2\sqrt{2} \omega \Omega^2 f(k) \sin \varphi - iD_{\nu}(k) \left( g k_z^2 + 2\sqrt{2} \omega \Omega \cos \left( k_y - k_x \right) \right) \right] \right\} = 0,$$

where $\bar{\nu} = 4\nu/3 + \mu$, $k^2 = \Sigma k_i^2$, $k_\perp^2 = k_x^2 + k_y^2$, $f(k) = \left( k_z \sin \varphi + \left( k_x + k_y \right) \cos \varphi \right)/\sqrt{2}$, $D_{\nu}(k) = \omega + i\nu k^2$, $\bar{D}_{\nu}(k) = \omega + i\nu k^2$, $D_{\kappa}(k) = \omega + i\kappa k^2$.

The power of the singularly perturbed dispersion equation (the leading, $k^8$, term involves a small factor $\nu^2 \bar{\nu} \kappa$) defines the number of roots. When all
dissipative coefficients equal to zero, Eq. (3) becomes a second order equation. So two of eight roots of Eq. (3) are regular in dissipative factors and describe propagating waves. The remaining six roots characterize the set of singular components including coexisting boundary layers. The question what components of motions propagate in the fluid body (only regular or regular and singular components which looks like interfaces in a fluid interior) is still open. The boundary of domains of propagating waves with the real frequency \( \omega \) existence depends on the ratio of wave and intrinsic rotation and buoyancy frequencies, the compressibility of the medium, and the geometry of the problem.

Solutions of Eq. (3) are further analyzed in the spherical coordinate system \((k, \Psi, \Theta)\) introduced in the wave number space \((k_x, k_y, k_z)\) by the relations \(k_x = k \sin \Theta \cos \Psi, \ k_y = k \sin \Theta \sin \Psi, \ k_z = k \cos \Theta\). Regular in dissipative factors solutions are written in a power series

\[
k_z^{(r)} = k_0 + \sum_{i,j,k=0}^{\infty} b_{ijk} k^i \nu^j \mu^k.
\] (4)

In the zero approximation all kinetic coefficients in Eq. (4) are equal to zero and solution for \(k_0\) has a form

\[
k_0 = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}; \quad \alpha = c^2 \left( N_c^2 \sin^2 \theta - \omega^2 + 4\Omega^2 F^2 \right)
\]
\[
\beta = 2\sqrt{2} \omega \Omega g \left( \sin \psi - \cos \psi \right) \cos \theta \cos \varphi;
\]
\[
\gamma = \omega^2 \left( \omega^2 - N^2 \right) + 4\Omega^2 \left( N_c^2 \sin^2 \theta - \omega^2 \right)
\] (5)

and the domain of existence of propagating waves of frequency \(\omega\) is defined by condition \(\beta^2 - 4\alpha\gamma \geq 0\) and determined by the inequality

\[
2\omega^2 \Omega^2 g^2 \sin^2 \theta \cos^2 \varphi \left( \sin \psi - \cos \psi \right)^2 \geq \frac{c^2 \left( 4\Omega^2 \left( N_c^2 \sin^2 \varphi - \omega^2 \right) - \omega^2 \left( N_c^2 - \omega^2 \right) \right)}{N_c^2 \sin^2 \theta - \omega^2 + 4\Omega^2 F^2}. \tag{6}
\]

More detailed calculations show that complete regular solution at the first approximation can be written as sum \(k_z^{(r)} = k_0 + k_1\) that is phase correction term depends on diffusivity only. Boundaries of frequency ranges depend on many
factors, such as the ratio of wave and intrinsic rotation and buoyancy frequencies, the compressibility of the medium and geometry of the problem.

4. Singular components of periodic motions

Singular components are analyzed on the plane whose direction of the normal is defined by the above mentioned angles $\psi$ and $\theta$ in local coordinate frame. Solution of the dispersion equation is a function, describing the dependence of the normal component of the wave number $k_n$ on two components, which are parallel to the plane. Approximate form of the dispersion equation, Eq. (3), defining only the main parts of the singular solutions in the local coordinate frame is

\[
\omega \kappa v^2 k_n^6 - iv \left[ \omega^2 v - \kappa \left( \left( N_c^2 - N^2 \right) \sin^2 \theta - 2\omega^2 \right) \right] k_n^4 +
\]

\[
+ \omega \left[ \nu \left( N_c^2 \sin^2 \theta - 2\omega^2 \right) + \kappa \left( \left( N_c^2 - N^2 \right) \sin^2 \theta - \omega^2 + 4\Omega^2 F^2 \right) \right] k_n^2 -
\]

\[
-i\omega^2 \left( N_c^2 \sin^2 \theta - \omega^2 + 4\Omega^2 F^2 \right) = 0
\]

There are no components of wave number, which are parallel to the contact plane in Eq. (7). Thus near the contact surface only solenoidal motions exist with $\text{div } v = 0$. Eq. (7) has three solutions with respect to variable $k_n^2$

\[
k_{n\kappa}^2 = -\frac{\omega}{\kappa} \left[ 1 + \varepsilon \frac{N_c^2 \sin^2 \theta}{\omega^2} - \varepsilon^2 \frac{N_c^2 \sin^2 \theta \left( N_c^2 \sin^2 \theta - \omega^2 \right)}{\omega^4} \right],
\]

\[
k_{n\nu\pm}^2 = \frac{1}{\nu} \left[ \omega_{\pm} - \omega + \varepsilon \frac{N_c^2 \sin^2 \theta}{\omega^2} \frac{N_c^2 \sin^2 \theta - \omega^2}{N_c^2 \sin^2 \theta - 2\omega \omega_{\pm}} \right]
\]

where $\varepsilon = \kappa/\nu = Sc^{-1}$ is the inverse Schmidt number (typically in a real fluid $\varepsilon << 1$) and

\[
\omega_{\pm} = \frac{N_c^2 \sin^2 \theta}{2\omega} \left[ 1 \pm \sqrt{1 + \frac{16\omega^2 \Omega^2 F^2}{N_c^4 \sin^2 \theta}} \right].
\]

Solution $k_{n\kappa}^2$ in Eq. (8) describes density boundary layer of a thickness $\delta_{\kappa \approx \sqrt{2\kappa/\omega}}$. The two other expressions $k_{n\nu\pm}^2$ in Eq. (9) describe two different viscous boundary layers with transverse scales $\delta_{\nu\pm} \approx \sqrt{2\nu/|\omega_{\pm} - \omega|}$. All three singular components exist at the same time and their relative thickness depends
upon parameter $\varepsilon$. When the rotation frequency is small $\Omega \ll N_c^2 \sin \theta / 4\omega F$ the viscous boundary layers have essentially different thickness as in this case $\omega_+ \approx N_c^2 \sin^2 \theta / \omega$, $\omega_- \approx 4\omega \Omega^2 F^2 / N_c^2$ and $\delta_{v+} = \delta_N \sqrt{2\omega^* / \left[ N_c^2 \sin^2 \theta - \omega^2 \right]}$, $\delta_{v-} = \delta_N \sqrt{2 / \omega^*} \left[ 1 + 4\Omega^2 F^2 / N_c^2 \right]$, where $\delta_N = \sqrt{v/N}$ is universal microscale. One of the velocity boundary layers with the thickness $\delta_{v+}$ is defined by viscosity, buoyancy frequency and compressibility effects. Parameters of the second layer with thickness $\delta_{v-}$ additionally depend upon the rotation frequency $\Omega$. In high frequency limit ($\omega^2 >> N_c^2 \sin^2 \theta$) thickness of both boundary layers tends to the Stokes scale $\delta_{v+} \approx \delta_v \approx \sqrt{2v/\omega}$.

Taking compressibility into account and disregarding rotation effects ($\Omega = 0$), we conclude from Eq. (6) that propagating three-dimensional acoustic gravity waves exist in two frequency bands $\omega \leq N_c$ and $\omega \geq N$. At the low frequency ($\omega \leq N_c$) they exhibit the properties of internal gravity waves. Their properties for $\omega \geq N$ approach the isotropic sound. Simultaneously with waves, two types of boundary layers with the characteristic thickness

$$\delta_{St} = \delta_N \sqrt{2 / \sin \Theta_\omega}, \quad \delta_i = \delta_N \frac{2 \sin \Theta_\omega}{\left| \sin^2 \Theta - \sin^2 \Theta_\omega \right|},$$

where $\delta_N = \sqrt{v/N}$, $\Theta_\omega = \arcsin \left( \omega / N \right)$, are formed on rigid boundaries. The first of them is similar to the periodic Stokes flow in a homogeneous fluid (Stokes, 1847), and the second, whose parameters depend both on the buoyancy frequency $N$ and on the speed of sound $c$, is specific for stratified media. The universal microscale $\delta_N = \sqrt{v/N}$ is common for both boundary layers. The thicknesses of the boundary layers also depend on the slopes of the waves and bounding surfaces.

The frequency band $\omega_- < \omega < \omega_+$ of the existence of inertial gravity waves in stratified rotating incompressible media is limited by the values

$$2\omega^2 \pm = N^2 + 4\Omega^2 \pm \sqrt{(N^2 + 4\Omega^2 \cos 2\varphi)^2 + 16\Omega^4 \sin^2 2\varphi},$$

which depend on the latitude of the observation point. Simultaneously with 3D inertial gravity waves, there are two types of boundary layers with the scales

$$\delta_{b\pm} = \delta_N \sqrt{2 / \left( \omega_\pm - \omega^* \right)}, \quad \omega_\pm = \frac{\sin^2 \Theta}{2\omega^*} \left[ 1 \pm \sqrt{1 + 16\omega^2 \Omega^2 F^2 / N^4 \sin^4 \Theta} \right] \omega^* = \frac{\omega}{N}. \quad (11)$$

Inertial acoustic waves in homogeneous fluid ($N = 0$) coexist with two separated boundary layers with the thickness
\[
\delta_{b \pm} = \sqrt{v/\Omega |F \pm \cos \Theta_o|},
\]

where \( \Theta_o = \arccos(\omega/2\Omega) \) is the slope of the propagation lines of the inertial acoustic waves to the horizon. One of these waves with thickness \( \delta_{b+} \) is an analogue of the known Ekman layer. Periodic flows have the properties of inertial and acoustic waves for \( \omega << \Omega \) and the opposite case, respectively.

Three-dimensional acoustic waves in a homogeneous fluid \( (N = \Omega = 0) \) are characterized by the dispersion \( \omega^2 = k^2 \left( c^2 - i\omega(4v/3 + \mu) \right) \). In this case two sets of boundary layers are joined in the united doubly degenerate Stokes layer with the thickness \( \delta_b = \sqrt{2v/\omega} \). Perturbations within this layer are transverse with zero divergence of the velocity; i.e., the fluid within it behaves as incompressible.

From the form of the dispersion of three-dimensional periodic perturbations in a homogeneous incompressible fluid, \( k^2 \left( \omega + \nu k^2 \right)^2 = 0 \), when \( (N = \Omega = \nabla \cdot u = 0) \), it follows that this medium is free of developed propagating waves. Doubly degenerated viscous boundary layer consisting of two periodic Stokes flows with thickness \( \delta_b = \sqrt{2v/\omega} \) is formed on the rigid oscillating boundary. It means that classical 3D Navier–Stokes equations both for compressible and incompressible fluids form ill-posed problem due to merging of boundary layers. The degeneration of Navier–Stokes equations for homogeneous fluids is removed by specific boundary conditions abolished one of the boundary layers (2D or axial-symmetric problems).

5. **Attached internal waves past horizontally moving strip**

Calculation of periodic waves beams show that singular components manifest itself near the contact rigid surface as set of boundary layers and inside the wave beam as fine components with their intrinsic spatial and temporal scales (Chashechkin et al., 2004). Here attached internal waves produced by a horizontally moving strip are discussed as example of 2D Navier–Stokes equations exact solution. In this case both the boundary conditions and governing equations are linearized.

In the simplest case when the source of disturbances is uniformly moving horizontal strip the set of governing equations, Eq. (1), is transformed into standard internal wave equation for stream function \( \Psi \) defining components of velocity \( u_x = \partial \Psi / \partial z, \ u_z = -\partial \Psi / \partial x \)

\[
\left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) + N^2 \frac{\partial^2}{\partial x^2} - v \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right)^2 \right] \Psi = 0
\]
with no-slip boundary conditions at the plane and on the strip of length \( a \)

\[
\frac{\partial \Psi}{\partial z} \bigg|_{z=0} = U \left( x + \frac{a}{2} - Ut \right) \Theta \left( \frac{a}{2} + Ut - x \right),
\]

\[
\frac{\partial \Psi}{\partial x} \bigg|_{z=0} = 0.
\]

and attenuation of all disturbances at infinity.

The solution of Eq. (13) is represented as the Fourier integral expansion:

\[
\Psi(x, z, t) = \int_{-\infty}^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} \left[ A_w(\omega, k) e^{ik_w^2(\omega, k)z} + B_i(\omega, k) e^{ik_i^2(\omega, k)z} \right] e^{ikx} dk d\omega
\tag{15}
\]

The roots of the dispersion equation, corresponding Eq. (13) are

\[
\omega^2\left(k^2 + k_z^2\right) - N^2k^2 + i\omega \nu \left(k^2 + k_z^2\right)^2 = 0
\tag{16}
\]

and include both regular in viscosity (corresponding to waves)

\[
k_w^2(\omega, k) = -k^2 + \frac{i\omega}{2\nu} \left[ 1 - \sqrt{1 + \frac{4i\nu k^2 N^2}{\omega^3}} \right],
\]

and singular in viscosity characterizing boundary layers

\[
k_i^2(\omega, k) = -k^2 + \frac{i\omega}{2\nu} \left[ 1 + \sqrt{1 + \frac{4i\nu k^2 N^2}{\omega^3}} \right].
\]

Substitution of Eq. (15) into the boundary conditions Eq. (14) leads to a system of algebraic equations for spectral component of amplitude

\[
A_w(w, k) = -A_i(w, k) = \frac{iU}{\pi k(k_w - k_i)} \sin \frac{ka}{2} \delta(\omega - kU).
\tag{17}
\]

Substitution of the solution of Eq. (17) into Eq. (15) and integration give the resultant expression for the stream function
\[
\Psi(x, z, t) = \frac{iU}{\pi} \int_{-\infty}^{\infty} \frac{1}{k} \sin \frac{ka}{2} e^{ik(x-Ut)} e^{ik_w(kU,k)z} - e^{ik_i(kU,k)z} \frac{k_w(kU,k) - k_i(kU,k)}{k} dk.
\]

(18)

From Eq. (18), it follows that the field of lee waves is transient ahead and stationary behind the source in the local reference frame (Bardakov and Chashechkin, 2004).

Visualization of exact solution, Eq. (18), for the vertical component of velocity and vorticity by the modified method of isopleths makes it possible to reveal not only the complete structure of transient leading and stationary attached internal waves, but also details of the fine structure of the boundary layer. Wave perturbations near the plate are more contrasting for the horizontal velocity than for the vertical one (Fig. 1). Moreover, the number of perturbation peaks in a single wave field turns out to be different for different wave components (one peak band for the horizontal component and two bands for the vertical one). The phase surface slope to the horizon is characterized by local frequency values.

(a) 

(b)

Fig. 1. Pattern of attached internal waves for horizontal (a) and vertical (b) components of velocity. Bright points on upper horizontal line indicate edges of the strip moving from left to right ($T_b = 6.28$ s, $a = 5.5$ cm, $U = 1$ cm/s, $\lambda = 6$ cm, $Fr = 0.18$, $Re = 550$).

The detailed structure of the module of field velocity within the boundary layer is shown as a magnified continuous-tone image in Fig. 2. Both leading and trailing edges of the plate incorporate singular perturbations with the vertical velocity oriented at first toward the fluid and, then, toward the plate. The edge singularities of the horizontal velocity are much less prominent. The thickness of Prandtl’s boundary layer (with a typical length scale of $\delta_u = \nu/U$) monotonously increases with distance from the leading edge the same way as in laminar flow of a homogeneous liquid. The boundary layer is detached from the trailing edge into the liquid. The complicated structure of boundary layers at a horizontally moving plate indicates that it is impossible to model the formation...
of attached internal waves near a real obstacle in terms of a set of singular mass or force sources.

\[ T_b = 14 \text{ s}, \quad \lambda = 14 \text{ cm}, \quad Fr = 1.12, \quad Re = 200. \]

The complex structure of a calculated flow pattern indicates that even on the strip a drag is produced due to emission of waves and viscous lost of energy. Calculated flow pattern fit in laboratory experiments when sensitive Schlieren instrument and markers are used for flow observation.

6. Conclusion

Analysis of complex questions put by F. Nansen shows that internal wave fields have a complex structure and include transient and attached waves and boundary singularities. Therefore, molecular effects play an important role in a stratified flow fine structure formation. Presented classification of 3D periodic motions in fluids generalizes classical Stokes scheme for the wave and accompanying periodic boundary layer. Two different types of viscous boundary layers on rigid boundaries supplement all kinds of waves. The first of them is similar to the periodic Stokes flow, the second one has no analogue in homogeneous fluid. Diffusivity effects are responsible for the formation of new kinds of boundary layers.

In the general case, the dynamics of hydrodynamic systems is determined by nonlinear interaction between all structural elements of flows including both regular (waves, vortices) and singular types (boundary layers and singular components in a fluid interior). In particular, variations in thickness and nonlinear interactions of boundary layers make the generation of internal waves possible even in the cases when the direct excitation is forbidden by the linear theory. Owing to large vorticity, interacting boundary layers may be effective generators of vortex motions. Instruments for experimental studies of fluid dynamics must resolve the fine structure of the smallest elements of flows.
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References


