Surface patterns on thin liquid films, reduced 2D-descriptions. Part I. Perfect fluids

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Abstract—In this paper we show, how reduced equations for thin fluid layers can be systematically derived from the basic hydrodynamic equations. In this part we consider perfect fluids with zero viscosity. The shallow water equations are derived and extended to the case of a bistable pressure. Numerical solutions show instabilities, coarsening and relaxational behavior versus stationary states. Finally, parametric excitation is included, and a rich spatio-temporal behavior of the surface structures is obtained.

Key-words: hydrodynamic waves, shallow water equations, reduced description

1. Introduction

More or less regularly structured surfaces on fluids or interfaces separating liquids (or a gas and a liquid) are found in nature as well as in technological applications on a large variety of length and time scales. Far from being complete, we mention some examples:

- Water waves caused by wind or sea quakes and land slides (tsunamis) (Kundu, 2004; Faber, 1995; Ward and Day, 2001).
- Localized excitations of the surface (solitons), i.e., on shallow water (Drazin and Johnson, 1989).
- Shear instabilities in clouds or multi-layer systems like the Kelvin-Helmholtz instability or the Rayleigh-Taylor instability (Kundu, 2004; Faber, 1995; Chandrasekhar, 1981).
- Surface deflections in the form of holes or drops of thin fluid films in coating or wetting processes (Bestehorn and Neuffer, 2001; Reiter, 1992).
• Creation and controlled growth of ordered structures in (nano-) technological applications (Kargupta and Sharma, 2001).
• Biological applications: Behavior of liquid films on leaves or of the tear film on the cornea of the eye; dynamics of thin blood layers; blood clotting.
• Films on the walls of combustion cells.
• Lubrication films in mechanical machines.

Surface patterns may occur due to several mechanisms. One mainly can distinguish between two cases: Patterns excited and organized by some external forces or disturbances (e.g., tsunamis), and those formed by instabilities. The latter may show the aspects of self-organization and will be in the focus of the present contribution (Haken, 2004; Cross and Hohenberg, 1993).

Complicated systems are controlled by a certain set of parameters, which can be accessed from outside. They are named “control parameters”. The states of the systems under consideration are described by state variables like temperatures, concentrations, velocity fields, etc. Changing one or more control parameters, the system may reach a certain critical point and an instability may occur. Then the old solution gets unstable and gives way to a qualitatively new behavior. Those instabilities can be regular (periodic) in space and/or time. In that way, new typical length or time scales are created by self-organization (Haken, 2004). On the other hand, they can also be homogeneous, stationary, or, the other extreme case, turbulent or chaotic.

The mathematical description of the systems listed above is more or less well known for a long time. Fluid motion is described by the Euler or Navier-Stokes equations, temperature fields by the heat equation, and chemical concentrations by some nonlinear reaction-diffusion equations. The location and spatio-temporal evolution of surfaces or interfaces can be computed by the kinematic boundary conditions, if the velocity of the fluid near the interface is known. All these equations can be coupled and provided with suitable boundary and initial conditions. In that way, rather complicated systems of nonlinear partial differential equations result. Even nowadays, in the age of supercomputers, their further treatment, especially in three room dimensions, remains still a challenge.

On the other hand, solving directly the basic equations, can be considered merely as another experiment. For these reasons, we wish to explore here other methods. In this and in the next contribution we shall describe how to derive reduced 2D descriptions of the fully 3D problems.

2. The shallow water equations

In this section we wish to derive briefly the equations describing the motion of a thin layer of a perfect fluid. They are called shallow water equations. Although
the derivation is standard (see, for instance, the textbooks (Kundu, 2004; Faber, 1995; Bestehorn, 2006), we wish to present it here, because it serves as a good example of how 2D equations can be found systematically from a 3D basic problem.

2.1. Potential flow

We consider an incompressible fluid. Its velocity field \( \mathbf{v}(\mathbf{r},t) \) is ruled by the Euler equations and the incompressibility condition:

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \mathbf{f},
\]

\[
\nabla \cdot \mathbf{v} = 0,
\]

where \( p(\mathbf{r},t) \) is the pressure, \( \rho \) is the (constant) density, and \( \mathbf{f}(\mathbf{r},t) \) denotes some external forces like gravity, etc. If the fluid is free of vorticity, \( \nabla \times \mathbf{v} = 0 \), the velocity can be expressed by a scalar potential \( \Phi(\mathbf{r},t) \):

\[
\mathbf{v}(\mathbf{r},t) = \nabla \Phi(\mathbf{r},t).
\]

From Eq. (1b) we get the Laplace equation:

\[
\Delta \Phi(\mathbf{r},t) = 0,
\]

which determines the velocity field along with suitable boundary conditions.

2.2. Kinematic boundary conditions

We assume the surface to vary on a large lateral scale compared to the liquid depth. In addition it should be writable as a (smooth) function \( h(x,y,t) \). If the fluid moves, the location of the surface is changed according to the kinematic boundary condition (kbc) (Fig. 1):

\[
\partial_t h = \mathbf{n} \cdot \mathbf{v} = (1 + (\nabla h)^2)^{-1/2} (v_z - v_x \partial_x h - v_y \partial_y h) \approx v_z - v_x \partial_x h - v_y \partial_y h,
\]

or, in terms of the potential

\[
\partial_t h = \partial_z \Phi - \partial_x \Phi \partial_x h - \partial_y \Phi \partial_y h.
\]

Here, \( \mathbf{v} \) (and \( \Phi \)) have to be evaluated at the surface \( z = h \), and a small surface slope \( (\nabla h)^2 \ll 1 \) is assumed. Inserting Eq. (2) into the Euler equations Eq. (1a) and evaluating them at the surface lead to the other kbc:
\[
\partial_t \Phi|_{z=h} + \frac{1}{2} (\nabla \Phi)^2_{z=h} = -g(h-h_0) - \frac{p(h)}{\rho}, \tag{5}\]

where we consider a constant gravity field in \(z\)-direction as external force, and a certain given pressure \(p(h)\) at the surface that will be specified later.

**Fig. 1.** (a) An (incompressible) fluid with a free and deformable surface located at \(z = h(x,y,t)\), on which a constant external pressure \(p_0\) is applied. (b) The location of a certain point of the surface changes if the fluid is in motion.

### 2.3. Scaling and small parameter

The crucial point in the derivation of the reduced equations is the different scaling used for the vertical and horizontal coordinates. Let \(h_0\) be the uniform depth of the motionless layer and \(\ell\) a certain horizontal length scale (wave length, spatial extension of a front, etc.), then we introduce the dimensionless variables

\[
x = \tilde{x} \ell, \quad y = \tilde{y} \ell, \quad z = \tilde{z} h_0, \quad t = \tilde{t} \tau, \tag{6}\]

and

\[
h = \tilde{h} h_0, \quad \Phi = \tilde{\Phi} \frac{\ell^2}{\tau}, \tag{7}\]

where \(\tau\) is a certain time scale that is left arbitrary for the moment. The different length scales define a small parameter

\[
\delta = \frac{h_0}{\ell} << 1, \tag{8}\]

which can now be used for a systematic expansion. In the dimensionless quantities the basic equations and boundary conditions read (we drop all tildes):
\[(\delta^2 \Delta_2 + \partial^2_{zz})\Phi = 0, \quad (9a)\]
\[\partial_t h - \delta^{-2} \partial_z \Phi|_{z=h} = - (\partial_x h)(\partial_x \Phi)_{z=h} - (\partial_y h)(\partial_y \Phi)_{z=h}, \quad (9b)\]
\[\partial_t \Phi|_{z=h} + G(h-1) = - p(h) - \frac{1}{2} \left( (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + \delta^{-2} (\partial_z \Phi)^2 \right)_{z=h}, \quad (9c)\]
\[\partial_z \Phi|_{z=0} = 0, \quad (9d)\]

with \(\Delta_2 = \partial_{xx} + \partial_{yy}\) as the 2D Laplacian, and the dimensionless gravitation number \(G = \frac{gh_0 \tau^2}{\ell^2}\).

### 2.4. Laplace equation

The next step is to solve the Laplace equation Eq. (9a), iteratively. Therefore, we expand

\[\Phi(r, t) = \Phi^{(0)}(r, t) + \delta^2 \Phi^{(1)}(r, t) + \delta^4 \Phi^{(2)}(r, t) + \ldots\]

and find from Eq. (9a) in the zeroth order of \(\delta\) that

\[\partial_{zz} \Phi^{(0)} = 0.\]

Because of the boundary condition, Eq. (9d), this can only be solved if it is independent of \(z\):

\[\Phi^{(0)}(x, y, t) = \Phi^{(0)}(x, y, t).\]

In the order \(\delta^2\) one then finds

\[\partial_{zz} \Phi^{(1)}(r, t) = - \Delta_2 \Phi^{(0)}(x, y, t),\]

which can be integrated twice:

\[\Phi^{(1)}(r, t) = - \frac{z^2}{2} \Delta_2 \Phi^{(0)}(x, y, t) + \varphi^{(1)}(x, y, t), \quad (10)\]

with an arbitrary function \(\varphi^{(1)}\). Up to the second order one gets

\[\Phi(r, t) = \Phi^{(0)}(x, y, t) + \delta^2 \left[ - \frac{z^2}{2} \Delta_2 \Phi^{(0)}(x, y, t) + \varphi^{(1)}(x, y, t) \right] + O(\delta^4). \quad (11)\]
2.5. The shallow water equations

Inserting Eq. (11) into the two kbc-s, Eqs. (9b)–(9c), and yielding up to the lowest order in $\delta$, the shallow water equations are

\[ \partial_t h = -h \Delta_2 \Phi^{(0)} - (\partial_x h)(\partial_x \Phi^{(0)}) - (\partial_y h)(\partial_y \Phi^{(0)}), \]  
\[ \partial_t \Phi^{(0)} = -G(h - 1) - p(h) - \frac{1}{2} (\partial_x \Phi^{(0)})^2 - \frac{1}{2} (\partial_y \Phi^{(0)})^2. \]

Now we have reached our goal to derive a 2D system starting from 3D fluid motion. Eqs. (12) constitute a closed system of partial differential equations for the evolution of the two functions $h(x, y, t)$ and $\Phi^{(0)}(x, y, t)$. Using Eq. (11), from the latter one finds the velocity field immediately (up to the order $\delta^2$).

2.6. Numerical solutions

Fig. 2 shows numerical solutions of the shallow water equations, left frame in 1D, right frame in 2D. In 1D, traveling surface waves can be seen clearly, which may run around due to the periodic boundary conditions in $x$. On the other hand, one can recognize a second wave with a smaller amplitude going to the left hand side. Both waves seem to penetrate each other without further interaction. The reason seems to be the smallness of the amplitude, which results in a more or less linear behavior.

Fig. 2. Numerical solutions of the shallow water equations. (a) Temporal evolution in one dimension. (b) A snapshot in 2D. Dashed contour lines mark troughs, solid ones correspond to peaks of the sea.
In the 2D frame, a snapshot of the temporal evolution of the surface is presented. The initial condition was chosen randomly. For numerical stability reasons, an additional damping of the form

\[ \tilde{\nu} \Delta_2 \Phi \]  

was added to the right hand side of Eq. (12b) in order to filter out the short wave lengths. This could be justified phenomenologically by friction and in the long time limit it leads to a fluid in rest, if only gravity acts.

3. Instabilities

To see if the flat film \( h = 1 \) is stable against small perturbations, one may perform a linear stability analysis. To this end it is convenient to introduce the (small) variable

\[ \eta(x, y, t) = h(x, y, t) - 1 \]  

in Eq. (12) and linearize with respect to \( \eta \) and \( \Phi \). The two resulting equations can be combined into one wave equation for \( \eta \) or alternatively for \( \Phi \):

\[ \partial_t \eta - G \Delta_2 \eta = 0, \]  

where we have assumed constant surface pressure. A solution of Eq. (15) is provided by

\[ \eta \sim e^{\lambda t + ikx}, \]

from which one finds the dispersion relation

\[ \lambda(k) = \pm i|k|\sqrt{G}, \]

for waves traveling with the constant phase velocity \( \pm \sqrt{G} \).

3.1. Damped wave equation

Since \( \text{Re}(\lambda) = 0 \), waves do neither growth nor decay in time, they are marginally stable. This changes if damping according to Eq. (13) is included. Then instead of Eq. (15) one finds a kind of telegraph equation having the dispersion relation

\[ \lambda(k) = -\frac{\nu k^2}{2} \pm i|k|\sqrt{G - \nu^2 k^2}. \]
Again waves, now with a slightly smaller phase speed, are the solution, but these waves are all decaying in time (if \( k \neq 0 \)), leading finally to a steady state with a flat surface at \( h = 1 \). So the initial state is now stable.

### 3.2. Laplace pressure and disjoining pressure

To discuss Eqs. (12) further, we have to elaborate a little on the dependence of the surface pressure on the depth \( h \).

The length scale of the surface structures is proportional to the depth of the fluid layer. If the films are very thin, we expect to have scales in the range or even well below the capillary length

\[
a = \sqrt{\frac{\Gamma}{G \rho}},
\]

(16)

where \( \Gamma \) denotes the surface tension. Then one has to take into account the additional pressure, which origins from the curvature of the surface, the so-called Laplace pressure. It reads for weakly curved surfaces

\[-q \Delta_2 h, \quad q > 0.\]

(17)

The dimensionless constant \( q \) is linked to the surface tension by

\[
q = \frac{h_0 \tau^2}{\ell^4 \rho} \Gamma.
\]

Thus we have to substitute \( p(h) \) in Eq. (12b) for the expression

\[p = p_0 - q \Delta_2 h,\]

(18)

which changes the dispersion relation to that of (damped) capillary waves

\[
\lambda(k) = -\frac{\nu k^2}{2} \pm i[k \sqrt{G + q k^2 - \nu^2 k^2}].
\]

Again, no instability with \( \text{Re}(\lambda) > 0 \) can occur.

Now assume, that the pressure \( p_0 \) depends also on the absolute value of \( h \) in some nonlinear non-monotonic fashion. This can be the case for very thin films, where van der Waals forces between the solid support and the free surface come into play (Israelachvili, 1992). But also in thicker films, this should be possible in non-isothermal situations, where the surface temperature and, therefore, the surface tension changes with the vertical coordinate (Marangoni effect).

If we take for instance the polynomial (Fig. 3a)
\[ p_0 = h(h-1)(h-2) \]  \hspace{1cm} (19)

linearization around \( h = 1 \) leads to the growth rates (Fig. 3b)

\[ \lambda_{12}(k) = -\frac{vk^2}{2} \pm i k^2 \sqrt{p'_0 + G + qk^2 - v^2k^2}, \]  \hspace{1cm} (20)

where

\[ p'_0 = \left. \frac{dp_0}{dh} \right|_{h=h_0} \]

and \( h_0 \) is the conserved mean thickness of the film. An instability occurs if the expression under the integral can be negative, i.e., for \( p'_0 + G < 0 \)

\[ \text{Fig. 3. (a) Non-monotonic generalized pressure with } p_0 \text{ from Eq. (19). Between the spinodals } h_a \text{ and } h_b, \text{ the flat surface is unstable and pattern formation sets in. The binodals } h_1 \text{ and } h_3 \text{ follow from a Maxwell construction. (b) Eigenvalues Eq. (20) for the supercritical case.} \]

This corresponds to the region of initial thickness, where the generalized pressure

\[ p_0 + G(h-1) \]  \hspace{1cm} (21)

has a negative slope (Fig. 3a). The two extrema, \( h_a \) and \( h_b \) are called spinodals. For \( h_a < h_0 < h_b \) the flat film is unstable with respect to infinitesimally small perturbations. The points \( h_1 \) and \( h_2 \) (the binodals) can be found for a general form of the pressure from a Maxwell construction. For the special form of Eq. (19) they coincide with the zeros of Eq. (21). Films having an initial depth in the two regions \( h_1 < h_0 < h_a, \ h_b < h_0 < h_3 \) are meta-stable. A finite disturbance is necessary to bring the system to an absolutely stable state, that consists of a structured surface. In these regions, pattern formation by nucleation is expected.
Fig. 4 shows a numerically determined time series of a random dot initial condition. The mean thickness $h_0$ was chosen in the unstable region on the right hand side of the Maxwell point $h_M$ of Fig. 3a. The formation shows clearly traveling waves in the linear phase, followed by coarsening to a large scale structure, in this case to one big region of depression, or a hole. This hole gets more and more symmetric, while the velocity decays due to the friction term. Finally, a steady state of a circular big hole remains.

Fig. 4. Time series from a numerical solution of Eq. (12) with damping Eq. (13) and bistable pressure (Eqs. (18), (19), and (21)). Coarsening dominates the non-linear evolution, and eventually, a stationary circular region of surface depression (a hole) remains. Parameters: $G=0.5$, $\nu=0.02$, $q=0.01$, $h_0=1.3$. Periodic boundary conditions in both horizontal directions have been used.
3.3. Parametric excitation of a thin bistable fluid layer

One way to replace the energy lost by a damping according to Eq. (13) is to accelerate the whole layer periodically in vertical direction. This was done first in an experiment by Michael Faraday in 1831. He obtained regular surface patterns mostly in the form of squares (Faraday, 1831).

Faraday patterns can be seen as a solution of the shallow water equations, if the gravity constant $G$ is modulated harmonically (Bestehorn, 2006):

$$G(t) = G_0 + G_1 \cos(\Omega t).$$  \hspace{1cm} (22)

A linear stability analysis leads to the Mathieu equation (now with $\nu=q=p_0=0$) (Abramowitz and Stegun, 1972):

$$\partial_{\tilde{t} \tilde{t}} \eta + (b^2 + 2a \cos(2 \tilde{t})) \eta = 0$$  \hspace{1cm} (23)

with

$$b^2 = \frac{4G_0 k^2}{\Omega^2}, \quad a^2 = \frac{2G_1 k^2}{\Omega^2},$$  \hspace{1cm} (24)

and the rescaled time $\tilde{t} = t\Omega/2$. The flat film is unstable if frequency and amplitude fall into certain domains, the so-called Arnold tongues (Fig. 5). There one usually finds squares for not too supercritical values.

![Fig. 5. The stability chart of the Mathieu equation (Eq. (23)) without damping, Arnold tongues. In region I, the pattern oscillates with the half of the driving frequency $\Omega$.](image)

Instead of presenting these results, we finally show a numerical solution of the full equations with $\nu, q, p_0$ having the same values as above, but now with
an additional periodic excitation (Fig. 6). Coarsening is still present, but now oscillating drops emerge in the form of stars. No time stable structure is found in the long time limit.
References